Multifrequency News and Stock Returns *

Laurent E. Calvet and Adlai J. Fisher

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*Calvet: HEC School of Management, 78351 Jouy-en-Josas Cedex, France, and National Bureau of Economic Research, Cambridge, MA 02138, USA, calvet@hec.fr. Fisher: Sauder School of Business, University of British Columbia, 2053 Main Mall, Vancouver, BC Canada V6T 1Z2, adlai.fisher@sauder.ubc.ca. We received helpful comments from an anonymous referee, Andrew Abel, Michael Brandt, John Campbell, Francesco Franzoni, John Geanakoplos, Ruslan Goyenko, Christian Lundblad, Bruno Solnik, Robert Stambaugh, Amir Yaron, Jessica Wachter, and seminar participants at CREST, Paris I Panthéon-Sorbonne, the Swedish Central Bank, the Wharton School, the 2005 UBC Summer Finance Conference, the 2005 World Congress of the Econometric Society, the 2006 Meeting of the Northern Finance Association, the 2006 Meeting of the Western Finance Association, and the 2006 Summer Finance Conference in Gerzensee. We are very appreciative of financial support provided for this project by the Agence Nationale de la Recherche, the HEC Foundation, the UBC Bureau of Asset Management, and the Social Sciences and Humanities Research Council of Canada.
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Abstract

Aggregate stock prices are driven by shocks with persistence levels ranging from daily intervals to several decades. To accommodate this diversity, we introduce a parsimonious equilibrium model with regime-shifts of heterogeneous durations in dividend news, and estimate specifications with up to 256 states on daily U.S. equity returns. The multifrequency equilibrium has significantly higher likelihood than the classic Campbell and Hentschel (1992) specification, while generating volatility feedback effects 10 to 40 times larger. Furthermore, Bayesian learning about volatility generates a novel tradeoff between skewness and kurtosis as information quality varies, complementing the traditional uncertainty channel (e.g., Veronesi, 1999). Economies with intermediate investor information best match daily stock returns.

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1. Introduction

Equity prices are driven by news with heterogeneous degrees of persistence, ranging from daily intervals to several decades. At short horizons, a variety of corporate and macroeconomic announcements affect investor forecasts of future cash flows and discount rates, and thus the dynamics of daily returns.\(^1\) For example, Campbell and Hentschel (1992) focus on how fluctuations in dividend news volatility impact endogenous stock prices; they show that innovations with a half-life of about six months contribute to matching daily returns. Equity markets also price diverse lower-frequency fundamentals such as demography (Abel, 2003), technological innovation (Pastor and Veronesi, 2005), and variations in consumption, dividends, and macroeconomic uncertainty (Bansal and Yaron, 2004; Lettau, Ludvigson and Wachter, 2004).\(^2\)

To accommodate this diversity, we develop a parsimonious asset pricing equilibrium model with shocks of heterogeneous durations. Our approach builds on recent developments in multifrequency econometrics. The Markov-switching multifractal (MSM) is a stochastic volatility model characterized by a small number of parameters but an arbitrarily large number of frequencies (Calvet and Fisher, 2001, 2002). Under this specification, volatility is hit by exogenous shocks with highly heterogeneous durations, which range from one day to more than a decade in empirical applications. MSM is sufficiently flexible to account for market conditions that change considerably over a long time span (Schwert, 1989). It also captures the outliers, volatility persistence and power variation\(^3\) of financial series, while permitting maximum likelihood estimation and analytical multi-step forecasting. MSM compares favorably with standard volatility models such as GARCH(1,1) both in- and out-of-sample (Calvet and Fisher, 2004; Calvet, Fisher, and Thompson, 2006).

The present paper embeds MSM, which has previously been limited to purely statistical applications, within a consumption-based asset pricing equilibrium. An Epstein-Zin consumer receives an exogenous consumption stream, and prices a flow of dividends with multifrequency Markov-switching.\(^4\) The economy is tightly parameterized and

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\(^1\) These announcements include macroeconomic conditions (Andersen, Bollerslev, Diebold, and Vega, 2004), weather news (Roll, 1984), analyst reports (e.g. Womack, 1996) and corporate activity (e.g., MacKinlay, 1997).

\(^2\) Other examples of financial news operating at different frequencies include the relatively high-frequency impact of liquidity uncertainty (Genette and Leland, 1990), intermediate contributions from political cycles (Santa-Clara and Valkanov, 2003), and low-frequency uncertainty regarding exhaustible energy resources.

\(^3\) Power variation relates to the behavior at small time scales of sums of powers of absolute values of returns. See Calvet and Fisher (2002), Barndorff-Nielsen and Shephard (2003) and Andersen, Bollerslev, and Diebold (2003).

\(^4\) Following Hamilton (1989), researchers have used regime-switching to help explain financial phenomena including stock market volatility, return predictability, the relation between conditional risk...
induces tractable expressions for equilibrium prices, return dynamics and filtered probabilities. The model generates volatility feedback, the property that upward revisions to anticipated future volatility tend to decrease current returns. Unlike previous Lucas tree economies (e.g., Bansal and Yaron, 2004; Lettau, Ludvigson and Wachter, 2004), higher volatility reduces prices for any level of the elasticity of intertemporal substitution.

Consistent with a multifrequency perspective, previous research has investigated volatility feedback at a range of different horizons. For example, French, Schwert, and Stambaugh (1987), Campbell and Hentschel (1992, hereafter “CH”), and Wu (2001) assess feedback effects in daily, weekly, and monthly data, while Pindyck (1984), Poterba and Summers (1985), Bansal and Yaron (2004), and Lettau, Ludvigson and Wachter (2004) emphasize volatility movements at the business cycle range and beyond.\textsuperscript{5} Intuition suggests that a multifrequency approach might prove useful in this context. High-frequency volatility shocks can help to capture the dynamics of typical day-to-day variations, while lower-frequency movements can generate the strong feedback required to fit the most extreme daily returns. The interaction of various frequencies thus seems important to understanding the effect of volatility fluctuations on stock returns. More broadly, the paper can be viewed as a first step towards bringing together branches of the lower-frequency macro-finance and higher-frequency financial econometrics literature.

We conduct structural estimation by maximum likelihood on an index of US equities over the period 1926-2003, and find that using six to eight volatility frequencies provides significant improvements relative to lower dimensional specifications. The model also improves on earlier specifications of single frequency news arrivals (CH), even though it uses fewer parameters.

The multifrequency equilibrium generates substantially larger feedback than previous research. For instance, CH find that feedback amplifies the volatility of dividend news by only about 1 to 2% depending on the sample; they attribute this result to the property of GARCH-type specifications that the volatility of volatility can only be large if volatility itself is high. With our MSM specification, feedback rises with the number of components and the likelihood function, increasing to between 20% and 40% for the preferred specifications. The multifrequency equilibrium model thus generates an unconditional feedback that is 10 to 40 times larger than in previous literature.\textsuperscript{6}

\begin{footnotesize}
\begin{enumerate}
\item General equilibrium investigation of volatility feedback was pioneered by Barsky (1989) in a two-period setting and Abel (1988) in the dynamic case. French, Schwert, and Stambaugh (1987) and CH use GARCH-type processes to show that ex-post returns are negatively affected by positive innovations in volatility. Bekaert and Wu (2000) provide further support for this hypothesis.
\end{enumerate}
\end{footnotesize}
Substantial return skewness is generally difficult to obtain in a full-information equilibrium with symmetric fundamentals. For this reason, earlier volatility feedback studies introduce predictive asymmetry (CH) or skewness (Wu, 2001) directly into the econometric specification of dividends. In this paper, we instead investigate whether asymmetry in returns can be modeled as the endogenous implication of imperfect investor information and learning. We generalize our setup to the case of noisy signals about volatility, and find that the sizeable feedback generated under full information is robust to changes in information quality.

Furthermore, signal precision controls a novel tradeoff between endogenous skewness and kurtosis. When information quality is poor, investors rely on dividend news to make inferences about the volatility state. They may learn quickly about volatility increases, because a single extreme fluctuation is highly improbable with low volatility. By contrast, learning about reduced risk must be slow because dividend news observations near the mean are a relatively likely outcome regardless of the true state. Thus, bad news about volatility incorporates quickly into price while good news trickles out slowly. As a consequence of this asymmetry, information quality has strong effects on both skewness and kurtosis; we find that intermediate signal precisions best capture the higher moments of daily stock returns.

Our study complements earlier research by Veronesi (2000) on how information quality affects stock returns. Whereas Veronesi considers learning about the latent drift in a two-state Lucas economy, our investors receive signals about an arbitrary number of dividend volatility components. By incorporating multiple shocks of heterogeneous durations, we obtain a structural learning model that is empirically relevant for higher-frequency daily stock returns.\footnote{Empirical implementation of learning models tends to focus on lower frequencies. For example, Veronesi (2004) calibrates to yearly returns and considers horizons ranging from twenty to two hundred years. Lettau, Ludvigson, and Wachter (2004) similarly consider highly persistent shocks with durations of about three decades. David (1997) and Brennan and Xia (2001) calibrate at a monthly frequency. Guidolin and Timmermann (2003) develop estimation and forecasting for a model of learning about the drift on a binomial lattice, and apply this to pricing options at a weekly frequency. At a monthly frequency, Turner, Startz, and Nelson (1989) and Kim, Morley, and Nelson (2004), consider learning about volatility in a two-state specification with feedback effects, where the signals that drive investor learning are not specified.}

Finally, we extend the multifrequency equilibrium to include long-run consumption risk, as in Bansal and Yaron (2004) and Lettau, Ludvigson, and Wachter (2004). With a relative risk aversion as low as $\alpha = 10$, the model gives a sizeable equity premium while maintaining substantial endogenous feedback. This demonstrates the robustness of our results to alternative assumptions about the dynamics of consumption growth.

Section 2 presents the asset pricing model and the equilibrium solution for a general
Markov structure. Section 3 specializes to a multifrequency volatility feedback setup. In Section 4, empirical results are provided for economies with full information. Learning economies are investigated in Section 5. Robustness checks and an extension to long-run consumption risk are presented in Section 6. All proofs are in the Appendix.

2. An Asset Pricing Model with Regime-Switching Dividends

This section develops a discrete-time consumption-based equilibrium with regime-shifts in the mean and volatility of dividend growth. Unlike traditional Lucas tree economies, the model generates a negative relation between volatility and prices for all preference parameters.

2.1. Preferences, Consumption and Dividends

We consider an exchange economy defined on the regular grid \( t = 0, 1, 2, ..., \infty \). As in Epstein and Zin (1989) and Weil (1989), the representative agent has isoelastic recursive utility

\[
U_t = \left\{ (1 - \delta)C_t^{\frac{1-\alpha}{\psi}} + \delta [E_t(U_{t+1}^{1-\alpha})]^{\frac{\psi}{1-\alpha}} \right\}^{\frac{\theta}{1-\alpha}},
\]

where \( \alpha \) is the coefficient of relative risk aversion, \( \psi \) is the elasticity of intertemporal substitution (EIS), and \( \theta = (1 - \alpha)/(1 - \psi^{-1}) \). When \( \alpha = \psi^{-1} \), the specification reduces to expected utility.

The agent receives an exogenous consumption stream \( \{C_t\} \). The log-consumption \( c_t = \ln C_t \) follows a random walk with constant drift and volatility:

\[
c_t - c_{t-1} = g_c + \sigma_c \varepsilon_{c,t},
\]

where the shocks \( \varepsilon_{c,t} \) are i.i.d. standard normal. This standard specification is consistent with the empirical evidence that consumption growth is approximately i.i.d. in postwar US consumption data (e.g., Campbell, 2003). In Section 6, we extend the model to allow small but highly persistent components in consumption, as in Bansal and Yaron (2004).\(^8\)

The volatility feedback literature suggests that aggregate stock prices decrease with the volatility of dividend news. When the stock market is a claim on aggregate consumption, the negative relation arises in equilibrium only for specific preferences. In the case of Epstein-Zin utility, volatility reduces prices only if \( \theta < 0 \), which requires that the EIS and risk aversion be either both strictly larger than unity (\( \alpha > 1 \) and \( \psi > 1 \)) or both strictly lower than unity (\( \alpha < 1 \) and \( \psi < 1 \)) (Bansal and Yaron, 2004; Lettau, \(^8\)Arguments in favor of long-run consumption risks are given by Bansal and Lundblad (2002), Bansal, Khatchatrian, and Yaron (2005) and Lettau, Ludvigson and Wachter (2004).)
Ludvigson and Wachter, 2004). While there is abundant evidence that $\alpha > 1$, the empirical validity of the EIS restriction has not been resolved. Attanasio and Weber (1993), Vissing-Jørgensen (2002), and Bansal and Yaron (2004) report estimates of $\psi$ larger than 1, while Campbell and Mankiw (1989), Campbell (2003), and Yogo (2004) find $\psi$ to be small and in many cases statistically indistinguishable from zero.

We resolve these difficulties by: 1) separating dividends from consumption, and 2) permitting that shocks to dividend volatility do not simultaneously impact consumption. Specifically, the log-dividend $d_t = \ln D_t$ follows a random walk with state-dependent drift and volatility:

$$d_t - d_{t-1} = \mu_d(M_t) - \frac{\sigma_d^2(M_t)}{2} + \sigma_d(M_t)\varepsilon_{d,t},$$

(2.2)

where $\varepsilon_{d,t}$ is i.i.d. standard normal and correlated with $\varepsilon_{c,t}$. The state $M_t$ is a first-order Markov vector with $k < \infty$ elements. The drift $\mu_d$ and volatility $\sigma_d$ are deterministic functions of the state $M_t$, and the Itô term $\sigma_d^2(M_t)/2$ guarantees that expected dividend growth $\mathbb{E}[D_t/D_{t-1}|M_t] = e^{\mu_d(M_t)}$ is controlled only by $\mu_d(M_t)$. We leave the exact specification of drift and volatility fully general in the rest of this section.

The model separates stock returns from aggregate consumption growth and the stochastic discount factor. This common assumption, (e.g., Campbell, 1996; Campbell and Cochrane, 1999), is consistent with the imperfect correlation between real consumption growth and real dividend growth. For instance, Campbell (2003) reports correlation estimates less than 0.5 in U.S. data, while Bansal and Yaron (2004) report and use in calibration a value of approximately 0.55. The disconnect between $d_t$ and $c_t$ is reasonable because corporate profits account for a small portion of national income. For instance, in US data corporate profits and personal consumption respectively account for approximately 10% and 70% of national income over the period 1929-2002. Consumption and dividend shocks should thus be correlated, but not identical.

### 2.2. Asset Pricing under Complete Information

We begin by considering that the agent directly observes the true state of the economy and has the full information set $I_t = \{(C_s, D_s, M_s); s \leq t\}$. This assumption holds for instance if agents observe the macroeconomic quantities determining the state or obtain $M_t$ by engaging into fundamental research.

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As shown in the Appendix, the aggregate consumption claim has a constant price/dividend ratio $P_c$ that satisfies

$$\frac{P_c}{1 + P_c} = \delta \mathbb{E}[(C_{t+1}/C_t)^{1-\alpha}] = \delta \exp \left[ (1 - \psi^{-1})g_c + (1 - \alpha)^2 \sigma_c^2/(2\theta) \right].$$

The equilibrium price decreases with volatility if and only if $\theta < 0$. 

---
The stochastic discount factor satisfies

\[ SDF_{t+1} = \delta' \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha}, \tag{2.3} \]

where \( \delta' = \delta \{ \mathbb{E}[(C_{t+1}/C_t)^{1-\alpha}] \}^{1/\theta} \), as shown in the Appendix. This expression is proportional to the stochastic discount factor obtained under expected utility \((\theta = 1)\), suggesting that the elasticity of intertemporal substitution affects the interest rate but not the price of risk.

We now turn to equilibrium pricing. The interest rate \( r_f = -\ln \mathbb{E}_t(SDF_{t+1}) \) is constant and obeys the familiar relationship:

\[ r_f = -\ln \delta + g_c/\psi - [\alpha + (\alpha - 1)/\psi] \sigma_c^2/2. \tag{2.4} \]

Consistent with earlier research (e.g. Hung, 1994), the equilibrium stock price is proportional to the current dividend, and the P:D ratio is controlled by the Markov state: \( P_t = Q(M_t)D_t \). The gross return on the stock

\[ \frac{D_{t+1} + P_{t+1}}{P_t} = \frac{D_{t+1} 1 + Q(M_{t+1})}{D_t Q(M_t)} \tag{2.5} \]

satisfies the Euler equation

\[ \delta' \mathbb{E} \left[ \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} \frac{D_{t+1} 1 + Q(M_{t+1})}{D_t Q(M_t)} \bigg| I_t \right] = 1. \]

The P:D ratio therefore solves the fixed-point equation

\[ Q(M_t) = \mathbb{E}_t \left\{ [1 + Q(M_{t+1})] e^{\mu_d(M_{t+1}) - r_f - \alpha \rho_{c,d} \sigma_c \sigma_d(M_{t+1})} \right\}, \tag{2.6} \]

where \( \rho_{c,d} \equiv \text{Corr}(\varepsilon_{c,t}, \varepsilon_{d,t}) > 0 \) denotes the constant correlation between the Gaussian noises in consumption and dividends. When the volatility process \( \{\sigma_d(M_t)\} \) is persistent, a large standard deviation of dividend growth at a given date \( t \) implies a low contemporaneous P:D ratio.\(^{10}\) High volatility thus feeds into low asset prices for any choices of the relative risk aversion \( \alpha \) and the EIS \( \psi \).

In empirical applications, the Markov vector \( M_t \) takes a finite number of values \( m_1, \ldots, m^d \). By fixed-point condition (2.6), the equilibrium P:D ratio can be computed numerically for every possible state \( Q(m^1), \ldots, Q(m^d) \). Econometric inference is also straightforward. While the investor observes the true state \( M_t \), we assume as in Campbell and Hentschel (1992) that the econometrician observes only excess returns and thus has the smaller information set \( I^0_t = \{ r_s \}_{s=1}^t \). By (2.5), the log excess return

\(^{10}\)Forward iteration implies \( Q(M_t) = \mathbb{E}_t \sum_{n=1}^{+\infty} \prod_{h=1}^n e^{\sigma_d(M_{t+h}) - r_f - \alpha \rho_{c,d} \sigma_c \sigma_d(M_{t+h})} \).
\[ r_{t+1} \equiv \ln((D_{t+1} + P_{t+1})/P_t) - r_f \] is determined by the price:dividend ratio and the realization of dividend growth:

\[ r_{t+1} = \ln \left( \frac{1 + Q(M_{t+1})}{Q(M_t)} \right) + \mu_d(M_{t+1}) - r_f - \frac{\sigma_d^2(M_{t+1})}{2} + \sigma_d(M_{t+1}) \xi_{d,t+1}. \] (2.7)

The likelihood function \( L(r_1, ..., r_T) \) then has the closed-form expression given in the Appendix.

3. Volatility Feedback with Multifrequency Shocks

We now specialize the above equilibrium to multifrequency dividend news. Calvet and Fisher (2001, 2002, 2004) develop the Markov-switching multifractal as a tractable model of heterogeneous persistence in financial series. We adopt this specification for dividend news, and examine the equilibrium implications.

3.1. Multifrequency Dividend News

We assume that the volatility of dividend news follows an MSM process, as is now explained. Given a state vector \( M_t \) with \( \bar{k} \) positive elements,

\[ M_t = (M_{1,t}; M_{2,t}; \ldots; M_{\bar{k},t}) \in \mathbb{R}_+^{\bar{k}}, \]

volatility is specified by

\[ \sigma_d(M_t) \equiv \tilde{\sigma}_d \left( \prod_{k=1}^{\bar{k}} M_{k,t} \right)^{1/2}, \] (3.1)

where \( \tilde{\sigma}_d > 0 \) is constant. Volatility is thus the product of \( \bar{k} \) distinct components or multipliers. The specification permits each component to evolve at a separate frequency. For example, the first component may have transitions measured in years or even decades, corresponding to low-frequency shocks in technology or demographics; medium-run components might represent business cycle fluctuations; and high-frequency components could capture liquidity or other transient effects. We need not specify the source of these fluctuations in advance. Instead, the number of components and their frequencies will be inferred directly from return data.

To maintain parsimony, we assume that the components of \( M_t \) evolve independently: i.e., \( M_{k,t} \) and \( M_{k',t'} \) are statistically independent if \( k \neq k' \). Further, the \( M_{k,t} \) are first-order Markov, and are statistically identical except for the difference in their time scales. For each component \( k \), a single parameter \( \gamma_k \) controls the persistence of innovations.

The components can be recursively constructed as follows. Given a value \( M_{k,t} \) for the \( k^{th} \) component at date \( t \), the next period multiplier \( M_{k,t+1} \) is either: (i) drawn from
a fixed distribution $M$ with probability $\gamma_k$, or (ii) left unchanged. The construction can be summarized as:

\[
\begin{align*}
M_{k,t+1} & \text{ drawn from distribution } M \quad \text{ with probability } \gamma_k \\
M_{k,t+1} & = M_{k,t} \quad \text{ with probability } 1 - \gamma_k.
\end{align*}
\]

The volatility components $M_{k,t}$ thus differ in their transition probabilities but not in their common marginal distribution $M$, which contributes to parsimony.

MSM can accommodate any distribution $M$ with positive support and unit mean. For simplicity, we choose it to be a binomial taking values $m_0 \in [1; 2]$ and $2 - m_0 \in [0; 1]$ with equal probability.

The transition probabilities $\gamma_k$ are tightly parameterized by

\[
\gamma_k = 1 - (1 - \gamma_k)^{(b^k - \bar{k})}.
\]

Calvet and Fisher (2001) introduce this specification through the discretization of a Poisson arrival process, and subsequent work demonstrates its empirical validity (Calvet and Fisher, 2004; Calvet, Fisher, and Thompson, 2006). Condition (3.2) implies that the transition probabilities grow approximately geometrically: $\gamma_k \sim \gamma_k^{b^k - \bar{k}}$. Thus, $\gamma_k$ controls the persistence of the highest frequency component and $b$ determines the spacing between components. These two parameters fully specify the set of frequencies $\{\gamma_k\}_{k=1}^\bar{k}$ regardless of the number of components $\bar{k}$.

One of the motivations of the paper is to understand the equilibrium implications of the above multifrequency volatility structure. We thus initially restrict the dividend growth rate to be constant:

\[
\mu_d(M_t) \equiv \bar{\mu}_d.
\]

This parallels the path taken in earlier research, where for example CH explore the equilibrium impact of single-frequency QGARCH dividend volatility dynamics with a constant drift. After investigating the constant drift specification in Sections 4 and 5, we extend the empirical implementation in Section 6 to include state-dependent drift in dividend news.

The volatility specification (3.1) has a number of appealing properties. Low-frequency multipliers deliver persistent and discrete switches, consistent with evidence of apparent non-stationarity in financial series (e.g., Schwert, 1989; Pagan and Schwert, 1990).\footnote{Although our model is strictly stationary, even very long-samples would be difficult to distinguish from a non-stationary process due to low-frequency switches. We view this as a convenient framework to model the low-frequency uncertainty that is present in financial data.} High-frequency multipliers give additional outliers through their direct effect on the tails of the dividend news process. Further, multiplicative interaction implies that total volatility can quickly switch from an extreme to a normal level, as has been observed in
equity data (e.g., Schwert, 1990b). We expect that these features of MSM will help to fit US stock returns over a long time span as well as to generate substantial volatility feedback.

3.2. Equilibrium Stock Returns

We now combine the general regime-switching economy in Section 2 with the MSM specification for dividend news. The equilibrium excess returns on the stock then satisfy

$$r_{t+1} = \ln \frac{1 + Q(M_{t+1})}{Q(M_t)} + \bar{\mu}_d - r_f - \frac{\sigma_d^2(M_{t+1})}{2} + \sigma_d(M_{t+1}) \epsilon_{d,t+1}. \quad (3.3)$$

In this equation, volatility feedback appears through the term $\ln([1+Q(M_{t+1})]/Q(M_t))$. Intuitively, an increase in a volatility component causes a decrease in the P:D ratio, which leads to a low realized return. An earlier version (Calvet and Fisher, 2005) uses a loglinearized return equation to confirm this logic and show: (i) the magnitude of the feedback due to a shift in an individual component is approximately proportional to the inverse of the persistence level $\gamma_k$; thus, lower frequency switches result in larger equilibrium feedback effects; (ii) the conditional return increases with the magnitude of the volatility components; again, lower-frequency components have a larger effect on the conditional mean.

The structural model implies tight specifications for the following observable time series: consumption, dividends, the riskless interest rate (2.4), the P:D ratio (2.6), and the excess stock returns (3.3). The economy is specified by preferences ($\alpha, \delta, \psi$), consumption ($g_c, \sigma_c$), dividends ($m_0, \gamma_k, b, \bar{\mu}_d, \bar{\sigma}_d$), and the correlation $\rho_{c,d}$. We now discuss the respective roles of these parameters.

The variables $g_c, \psi, \delta$ appear only in the interest rate equation. For any desired values of $g_c, \psi$, and the other parameters ($\alpha$ and $\sigma_c$), we can choose $\delta$ to match an arbitrary fixed interest rate. Without loss of generality, we will therefore calibrate the interest rate to its long run-value $\bar{r}_f$. In our empirical applications, the implied $\delta$ always takes annualized values in the 0.96 − 0.995 range, which seems reasonable.

Following the literature (e.g., CH), our empirical work calibrates the mean P:D ratio to a plausible long-run value: e.g.,

$$\text{EQ}(M_t) = \bar{Q}. \quad (3.4)$$

This guarantees that volatility feedback estimates do not arise from a counterfactually high share of dividends in stock returns. Since $\text{EQ}(M_t)$ monotonically decreases in $\alpha \sigma_c \rho_{c,d}$, the restriction on the average P:D ratio identifies $\alpha \sigma_c \rho_{c,d}$, conditional on the values of the five dividend parameters. By taking values of $\sigma_c$ and $\rho_{c,d}$ from consumption and dividend data, we can then infer an implied value of risk-aversion $\alpha$. Given our
standard setup, we anticipate that matching the equity premium in long-run data will require relatively large risk aversion, as suggested by the Hansen and Jagannathan (1991) bound. To demonstrate that our base results in Sections 4 and 5 are robust to lower values of $\alpha$, we introduce in Section 6 an extension with long-run consumption risk.

Given calibrated values of the risk-free rate and average P:D ratio, excess stock returns are specified by:

$$(m_0, \gamma_k, b, \bar{\mu}_d, \bar{\sigma}_d) \in \mathbb{R}^5.$$  

The parameters $\bar{\mu}_d$ and $\bar{\sigma}_d$ are important variables in any consumption-based asset pricing model, while $m_0, \gamma_k, \text{ and } b$ are specific to the MSM specification. To facilitate comparisons with earlier literature, we calibrate $\bar{\mu}_d$ and $\bar{\sigma}_d$ to commonly used values derived from aggregate dividend data. We then estimate the three remaining MSM parameters by maximizing the likelihood of daily excess returns.

This approach is valid if long-run moments such as $\bar{\mu}_d$ and $\bar{\sigma}_d$ can be adequately captured by aggregate dividend data. On the other hand, the high-frequency dynamics of dividend news volatility may be better reflected in stock returns than in dividends themselves. This structural approach to inferring dividend news parameters from excess returns data is consistent with much of the volatility feedback literature (e.g., CH; Wu, 2001).

Unlike existing asset pricing literature, the MSM setup can accommodate an arbitrarily large number of volatility frequencies while retaining a small and constant number of parameters. This will allow us to estimate a fully specified structural model of volatility feedback at a daily observation frequency.

4. Empirical Results with Fully Informed Investors

We now investigate the performance of the multifrequency equilibrium model on a long US equity series.

4.1. Excess Return Data

We estimate the multifrequency equilibrium model on daily excess returns of a US equity index from January 1926 to December 2003. As in CH, the index is constructed by combining the Schwert (1990a) daily index from 1926-1963 with CRSP value-weighted returns from 1963 onwards, and subtracting a daily risk-free rate imputed from 30-day Treasury bills. The entire period contains 20,765 observations (“Full Sample”). As is common in previous literature (e.g., CH), we also report results for the period beginning in 1952, because it corresponds to a change in interest rate regime with the Fed-Treasury Accord. This sample contains 13,109 observations (“Postwar Sample”).
Figure 1 shows the data, demonstrating the thick tails, low-frequency cycles, and negative skewness that are widely recognized characteristics of aggregate stock returns. To further indicate how conditions change across different periods in the long span of the data, Table 1 shows moments of the excess return series for four evenly spaced subperiods of each sample. These vary substantially, consistent with the findings of Schwert (1989) and Pagan and Schwert (1990). The data thus contain high-frequency variations as well as substantial movements at low frequencies.

4.2. Maximum Likelihood Estimation and Volatility Feedback

We begin by investigating the model under a single set of calibrated values. We choose the average P:D ratio $Q = 25$, similar to the long-run estimates reported by Campbell (2003) and Fama and French (2002). For the standard deviation of real dividend growth, we initially set $\sigma_d = 0.70\%$ per day (about 11% per year). This value is in the middle of the range of US historical estimates and is also very close to values used in earlier consumption-based calibrations (e.g. Campbell and Cochrane, 1999; Bansal and Yaron, 2004). To acknowledge the considerable uncertainty surrounding $\sigma_d$ in the literature and the importance of this parameter in any study of volatility feedback, we examine $\sigma_d$ closely in subsequent sensitivity analysis, using values ranging from 7.75% to 12.4% annually. Finally, we choose $\mu_d$ to (i) approximate the long-run average dividend growth rate and (ii) give a reasonable equity premium. Our initial calibration sets excess dividend growth $\mu_d - r_f$ to 0.5 basis points (bp) per day, or about 1.2% per year. Given an annual risk free rate of 1%, this implies $\mu_d = 2.2\%$ per year, similar to the values used in Bansal and Yaron (2004) and Lettau, Ludvigson and Wachter (2004). These values imply an average equity premium of about 4.6% annually. Our base calibration thus ensures reasonable values for the real dividend growth mean and variance, P:D ratio, and equity premium.

Table 2 reports maximum likelihood (ML) estimation results for the MSM volatility parameters $(m_0, \gamma_{\bar{k}}, b)$, conditional on the calibrated parameters. For the full sample in Panel A, and the postwar sample in Panel B, we consider a range of volatility components $\bar{k}$ varying from 1 to 8 by rows. The first row ($\bar{k} = 1$) of each panel corresponds to a standard regime-switching model with only two possible volatility states.

Examining the value of the likelihood function as $\bar{k}$ increases, we see the benefits of a multifrequency specification. In going from one to two volatility components, the log likelihood increases by over 3,000 points in the full sample, and over 700 points in the

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postwar sample. Since this requires adding only one additional parameter (from two to three), the increase in likelihood is large by any standard model selection criterion.\footnote{For example, using the Akaike Information Criterion (AIC) or the Bayesian Information Criterion (BIC) the necessary increase in likelihood to justify one additional parameter would be less than five points for either sample size.} Increasing the number of frequencies from two to three raises the log likelihood by an additional 1360 points in the full sample, and 260 points in the postwar sample, but does not increase the number of parameters.

Even after adjusting for the sample size, the proportionate increase in the likelihood \( (\Delta \ln L) / T \) is considerably larger for the full sample than for the postwar sample. This is not surprising for two reasons. First, over a longer time period we expect regime-changes to play an increasingly important role. Second, Table 1 showed that average return volatility is larger in the full sample (1.1% per day) than the postwar sample (0.85% per day). Thus, volatility feedback effects should be more beneficial in the full sample, and this is reflected in the proportionately larger increase in likelihood as volatility components are added.

Substantial increases in the likelihood continue, without adding additional parameters, throughout the set of \( \bar{k} \) that we examine. The likelihood function appears monotonically increasing and concave in \( \bar{k} \), and tends to flatten markedly by the time we reach the maximum value of \( \bar{k} = 8 \).

Since MSM specifications with different \( \bar{k} \) are nonnested, we can assess significance in the log-likelihood differences using the Vuong (1989) test and a HAC-adjusted version proposed in Calvet and Fisher (2004). For each sample and each value of \( \bar{k} = \{1, ..., 7\} \), we test the hypothesis that the specification has a higher likelihood than the model with \( \bar{k} = 8 \) components. In the postwar sample using the standard Vuong test, we reject this hypothesis at the 5% level for \( \bar{k} < 6 \), and at the 10% level in all cases. For the HAC adjusted version, the test is significant at the 5% level for \( \bar{k} < 4 \), at the 10% level for \( \bar{k} = 4 \), and marginal for higher values. In the full sample, the standard Vuong test is significant at the 1% level for all models. The HAC adjusted version is significant at the 1% level for \( \bar{k} < 7 \), and at the 5% level for all specifications. The statistical significance of the preference for the model with \( \bar{k} = 8 \) components is thus moderate in the postwar sample, and strong in the full sample.

The parameter estimates in Table 2 show reasonable patterns as \( \bar{k} \) varies. In both samples, the multiplier \( m_0 \) decreases monotonically as components are added. This is intuitively sensible since as \( \bar{k} \) grows, each individual component needs to do less work in explaining aggregate volatility fluctuations. The switching probability \( \gamma_{\bar{k}} \) of the highest frequency component is fairly stable across specifications, while the spacing parameter \( b \) tends to fall with \( \bar{k} \). These results imply that the highest frequency volatility shocks have durations of approximately 15 to 30 days. Adding volatility components tends to
tightly the intrafrequency spacing $b$ as well as to extend the low frequency range of volatility variations.

To demonstrate this last result, we report the largest duration in annual units (LFY) in the next to last column of Table 2. We see that for low $\bar{k}$, LFY tends to be under a year. For the preferred specifications with 6 to 8 components, the lowest frequency shocks are in the range of 10 to 20 years, a potentially reasonable value for technology or demographic changes. Since the frequency parameters driving the LFY statistics are estimated directly through the equilibrium likelihood function, this finding provides additional support to earlier specifications emphasizing low-frequency shocks. The estimated durations of LFY are roughly consistent with the durations assumed by Lettau, Ludvigson and Wachter (2004) and suggest that the approximately 2.5 year half-life of shocks in Bansal and Yaron (2004) may even be somewhat conservative.

Table 2 also reports statistics of the first four moments for each specification. The mean return of approximately 1.9 bp per day is close to the values of 2.2 in the full sample and 2.3 in the postwar sample. This is not surprising, since we have chosen $\bar{\mu}_d$ and the average P:D ratio to approximately match the long-run equity premium reported in the literature. The mean excess return does not vary substantially across specifications with different $\bar{k}$, since the volatility shocks do not affect consumption and thus should not affect expected returns. The volatility of excess returns tends to increase with $\bar{k}$. In the postwar sample, the specifications with large $\bar{k}$ have standard deviations of 0.8% per day, which is similar to the empirical value of 0.85%. In the full sample, the unconditional standard deviation is larger for $\bar{k} = 8$ at 0.83% per day, but still substantially smaller than the approximately 1.1% in the data. The model consistently captures a moderate degree of negative skewness, but the data is more negatively skewed. On the other hand, specifications with large $\bar{k}$ seem to have high kurtosis relative to the data. We will later use investor learning to endogenously attenuate kurtosis and enhance negative skewness.

The unconditional volatility feedback

$$FB = \frac{Var(r_{t+1})}{Var(d_{t+1} - d_t)} - 1$$

is presented in the last column of Table 2. Feedback increases as components are added: for the best performing models with $\bar{k} \geq 6$, it amplifies the variance of dividends by about 40 – 50% in the full sample and 11 – 30% in postwar data. These numbers are substantially larger than the 1 to 2% reported by CH, and we investigate in the next subsection whether this difference continues to hold in our longer samples.
4.3. Comparison with CH

The Campbell and Hentschel (1992) specification, described in the Appendix, provides a good comparison for our approach. First, CH also use endogenous feedback to generate restrictions on excess stock returns. They assume QGARCH dividend news and a linear pricing rule for volatility, which can be approximately reconciled with an equilibrium setup comparable to ours. Second, CH similarly address feedback effects in daily data over a very long sample, which is ambitious since departures from normality are most pronounced at shorter observation intervals, and changing economic conditions are more important over longer time spans (Schwert, 1989; Pagan and Schwert, 1990). More recent studies (e.g., Wu, 2001) often focus on lower frequency data over shorter time spans. Third, like CH we restrict dividend news to be symmetric and conditionally normal, which requires endogenous feedback to play a critical role in matching higher moments. By contrast, Wu (2001) allows correlation between dividend news growth and volatility, permitting exogenously skewed dividend news. Fourth, both CH and our model are relatively parsimonious. CH allow seven free parameters, while our specification imposes stronger economic restrictions and has only three free parameters. Finally, both our model and CH permit convenient ML estimation, which further facilitates comparison.

Table 3 reports ML estimation results for the CH model on both samples. Panel A gives parameter estimates, which are comparable to those found in the original CH study. The implied half-life of a volatility shock is about six months on both samples, which is again consistent with CH (1992). We also calculate the magnitude of volatility feedback by using the formula given in the Appendix. As in the original study, feedback contributes between 1-2% of unconditional variance, and is thus small relative to the MSM equilibrium.

Panel B compares in-sample fit of the CH model to the multifrequency specification with \( \bar{k} = 8 \) volatility components. Although the MSM equilibrium has four fewer free parameters, its likelihood is over 400 points larger in the full sample, and almost 200 points larger in the postwar sample. We adjust for the number of parameters by calculating the Bayesian Information Criterion (BIC) statistic for each specification, and assess significance using the Vuong (1989) test and the HAC-adjusted version of Calvet and Fisher (2004). The difference in likelihood is significant in both samples. MSM equilibria with \( \bar{k} > 4 \) volatility components have higher likelihood than CH in the full sample, and in the postwar sample MSM specifications with \( \bar{k} > 3 \) have higher likelihood. This confirms that the full-information multifrequency equilibrium generates large feedback effects and performs well in-sample relative to an important benchmark.
4.4. Conditional Inference

In the full-information framework, investors directly observe the volatility state $M_t$, but the empiricist makes inferences based only on excess returns $I_t^0 \equiv \{r_s; s \leq t\}$. The Appendix discusses how to calculate filtered probabilities $\hat{\Pi}_t^j \equiv \mathbb{P}(M_t = m^j | I_t^0)$ as well as the smoothed probabilities $\hat{\Psi}_t^j \equiv \mathbb{P}(M_t = m^j | I_T^0)$ for $j \in \{1, ..., 2^k\}$. The filtered probabilities are useful for forecasting, while their smoothed versions allow the most informative ex post analysis of the data.

Figure 2 displays the corresponding marginals of each component when $\bar{k} = 8$. Specifically, let $\hat{\Pi}_t^{M(k)} \equiv \mathbb{P}(M_{k,t} = m_0 | I_t^0)$ and $\hat{\Psi}_t^{M(k)} \equiv \mathbb{P}(M_{k,t} = m_0 | I_T^0)$ respectively denote the filtered and smoothed probabilities that volatility component $k \in \{1, \ldots, \bar{k}\}$ is in a high state.

Filtered probabilities on the left side of the figure show sensible patterns. For the lowest frequency $k = 1$, the probability rises over time from 0.5 to hover around 0.75 until the 1987 crash, and then jumps immediately to almost 1.0. The model thus attributes a portion of the very large price drop to an increase in low-frequency volatility. By contrast, when a smaller but still substantial drop in price of about $-8\%$ occurs just after 1955, probabilities about the first $k = 1$ and second $k = 2$ components move little, but the third component $k = 3$ jumps upwards substantially. When a similar size price drop occurs in the early 1960’s, the third volatility component already appears to be in the high state, so it cannot absorb the shock. Filtered probabilities about components $k = 4$ and higher thus increase. In general, the cycles have shorter durations as $k$ increases. For low values of $k$, the conditional distribution of the volatility state spends considerable time at the extreme values of zero and one. By contrast, at high frequencies probabilities move up and down rapidly, but rarely reach their boundaries. More refined inference is obtained by conditioning on all returns in the right-hand side of the figure. The smoothed marginals move less frequently but in larger increments, and spend more time near the boundaries of zero and one.

In Figure 3, we use the filtered probabilities to compute the one-step-ahead conditional mean and variance of returns. As implied by the equilibrium conditions discussed in Section 3, these are positively correlated, showing small peaks in the early 1970’s with higher levels in 1987 and around 2000. Recent literature (Ghysels, Santa-Clara, and Valkanov, 2005; Lundblad, 2005) finds empirical support for this type of relation.

The asset-pricing literature emphasizes that the market discount rate exhibits small and persistent variations through time. Feedback models focus on cyclical variations in dividend news volatility as a possible source of these fluctuations.\footnote{Other explanations include investor heterogeneity, habit-formation, or prospect theory. See Campbell (2003) for a recent review.} While our mul-
tifrequency volatility specification permits multiple sources of volatility fluctuations in accord with economic intuition, one might worry that this would lead to a conditional mean that is “too variable” or “too jumpy.” Figure 3 shows that this is in fact not the case. The conditional discount rate moves slowly because it is dominated by the most persistent volatility components.

4.5. Return Decomposition

We now develop an ex post decomposition of U.S. equity returns into a conditional expectation, feedback innovation, and dividend news. At time \( t + 1 \) or later, the fully-informed investor observes the excess return \( r_{t+1} \) and can implement the decomposition

\[
r_{t+1} = \mathbb{E}(r_{t+1}|M_t) + [\mathbb{E}(r_{t+1}|M_t, M_{t+1}) - \mathbb{E}(r_{t+1}|M_t)] + \sigma_d(M_{t+1}) \epsilon_{d,t+1}.
\]

This separates the realized return into: 1) its expected value at time \( t \); 2) the innovation due to the volatility feedback; and 3) the multifrequency dividend news.

The empiricist with smaller information set \( I_T^0 \) can derive an analogous but less precise decomposition. Specifically, the Appendix shows that the relation \( r_{t+1} = \mathbb{E}(r_{t+1}|I_T^0) \) implies

\[
r_{t+1} = \mathbb{E}\hat{\Psi}_t r_{t+1} + \left( \mathbb{E}\hat{\Psi}_{t+1} - \mathbb{E}\hat{\Psi}_t \right) r_{t+1} + \hat{e}_{d,t+1},
\]

where

\[
\hat{e}_{d,t+1} \equiv \mathbb{E}[\sigma_d(M_{t+1}) \epsilon_{d,t+1} | I_T^0]
\]

is the ex post estimate of realized dividend news. By the law of iterated expectations, \( \hat{e}_{d,t+1} \) has mean zero.

We implement the ex post decomposition in Figure 4. The top panel (4A) illustrates the excess return series \( \{r_t\} \), and the remaining panels show consecutively the three smoothed terms of (4.1): conditional return, volatility feedback, and dividend news. We examine these successively.

The smoothed conditional return in Panel 4B shows small persistent variations, very much like the ex ante conditional return in Figure 3. By contrast, the smoothed feedback in Panel 4C appears in intermittent bursts. On most days it is small, but its occurrences coincide with the most substantial variations in the series, and on these days it contributes a large portion of realized returns. These features are consistent with the intuition that low-frequency volatility changes are infrequent but have a large price impact. In particular, this ex post analysis attributes over half of the 1987 crash to volatility feedback.

Finally, in Panel 4D, the residual \( \hat{e}_{d,t+1} \) is the filtered version of a symmetric MSM process. We calculate its sample moments, and find a variance of 0.635, skewness coefficient \(-0.123\), and kurtosis 8.00. Relative to the actual return data, the residual variance
is approximately 16% smaller, skewness is 88% smaller, and leptokurtosis is 79% smaller. These findings suggest that endogenous volatility feedback plays an important role in explaining the higher moments of returns in our sample.

4.6. Alternative Calibrations

We now examine the robustness of the results to alternative calibrations of the main economic parameters. We specifically explore different values of the calibrated mean \( \bar{\mu}_d \) and standard deviation \( \bar{\sigma}_d \) of dividend growth, and of the average P:D ratio \( \bar{Q} \). For each alternative calibration, we reestimate the volatility parameters \( (m_0, \gamma_k, b) \) of the MSM equilibrium with \( k = 8 \) components using the constrained likelihood function.

We also assess how likely each model is to generate return moments similar to the data. For each specification, we simulate 1,000 paths of the same length as the data and calculate the fraction of paths for which a given statistic (mean, variance, skewness, kurtosis) exceeds the corresponding empirical moment.

The results of the estimations and simulated \( p \)-values are reported in Table 4. The first four rows in each panel hold \( \bar{\sigma}_d = 0.7\% \) per day, but allow varying combinations of \( \bar{Q} \in \{25,30\} \) and \( \bar{\mu}_d - r_f \in \{0.5,1.0\} \) bp per day, or 1.2% and 2.4% on an annual basis. Increasing the P:D ratio \( \bar{Q} \) leads to a lower equity premium (through the Gordon growth formula), which decreases feedback. To partially compensate, the estimates of \( m_0 \) and \( b \) increase slightly, implying somewhat larger and more persistent volatility shocks. Overall, the effect of the lower equity premium dominates, and the feedback measure \( FB \) declines as the average P:D ratio increases. Similarly, raising the dividend growth rate \( \bar{\mu}_d \) augments the equity premium and the magnitude of price shocks, \( m_0 \) and \( b \) decrease to partially compensate, and the net effect on feedback is positive. The likelihood tends to increase with \( \bar{\mu}_d \) and decrease with \( \bar{Q} \), and thus favors specifications with a larger feedback.

The final three rows in each panel hold \( \bar{\mu}_d \) and \( \bar{Q} \) constant at their original values. Instead, dividend volatility \( \bar{\sigma}_d \) increases to 0.8% per day (approximately 12.4% annually), or decreases to 0.6% or 0.5% per day (9.3% and 7.7% annually). These changes in dividend volatility have a considerable impact. When \( \bar{\sigma}_d \) is low, the model needs to generate large feedback in order to better approximate the volatility of excess stock returns. Generally, this tends to favor larger and more persistent volatility shocks. In the full sample, the model with \( \bar{\sigma}_d = 0.5 \) generates extremely large kurtosis (1481) and very persistent shocks (LFY = 160 years); the estimated feedback is over 150%. Larger values of \( \bar{\sigma}_d \) generate more moderate feedback in the range of 30 – 50%, and have substantially higher likelihood. In the postwar period where average return volatility is not as high, the model better accommodates low values of \( \bar{\sigma}_d \). The highest likelihood in the postwar data occurs in the base case where \( \bar{\sigma}_d = 0.7 \).
The simulated p-values show that the model generally captures well the mean return. The empirical skewness is more negative than expected under the model, but typically within the range of values that can be generated with our sample sizes. In the postwar period, the return volatility is lower than implied by the model, but the p-values are not significant. In the full sample, the p-values for the second moment are significant at the 5% level. Finally, all of the full sample specifications, and some of the postwar specifications, have significant p-values for a kurtosis that is too large. In summary, large feedback effects are robust across different calibrations, but kurtosis can become excessive when dividend volatility is very low.

A desirable improvement would be to consider a variant of the current model that generates lower kurtosis and stronger negative skewness in returns. CH amplify negative skewness by incorporating exogenous predictive asymmetry in the dividend news process. (Predictive asymmetry is the property that negative innovations are associated with higher future volatility than positive innovations of same magnitude.) When exogenous dividend news have this feature, volatility feedback is immediately asymmetric, giving stronger negative skewness in returns.

Our MSM specification for dividend growth has no predictive asymmetry, and incorporating it would certainly enhance negative skewness. On the other hand, this modification would not address the issue of high kurtosis, and would be more an econometric approach than an equilibrium explanation. In the next Section, we maintain the symmetric MSM dividend process, and instead show that learning about stochastic volatility can be a powerful method of endogenously amplifying negative skewness and moderating kurtosis.

5. Learning About Volatility and Endogenous Skewness

This section shows that learning about stochastic volatility provides a substantial source of endogenous skewness not previously identified in the literature. We assume that Bayesian investors receive imperfect signals about the state of the economy, which is a reasonable reduced form if fundamental research is costly. Signal quality controls a tradeoff between endogenous skewness and kurtosis: as information quality deteriorates, returns exhibit less kurtosis and more negative skewness. We show that (i) the size of the volatility feedback effect is not highly sensitive to the learning environment, and (ii) intermediate information levels best capture the higher moments of stock returns.

5.1. Investor Information and Stock Returns

Investors observe every period consumption, dividends, and noisy observations of the volatility components:

\[ \delta_t = M_t + \sigma_{\delta} z_t, \]  

(5.1)
where $\sigma_\delta$ is a nonnegative scalar, and $z_t \in \mathbb{R}^k$ is an i.i.d. vector of independent standard normals. This specification nests the full information case ($\sigma_\delta = 0$). The information set $I_t = \{(C_t, D_t, \delta_t); t' \leq t\}$ generates a conditional probability distribution $\Pi_t$ over the volatility states $\{m^1, ..., m^d\}$, which can be computed recursively.

The stochastic discount factor depends only on consumption and is thus the same as in the full information economy. The price:dividend ratio

$$Q(\Pi_t) = \mathbb{E} \left[ \sum_{i=1}^{\infty} \delta^i \left( \frac{C_{t+i}}{C_t} \right)^{-\alpha} \frac{D_{t+i}}{D_t} \mid I_t \right].$$

(5.2)

is the conditional expectation of exogenous variables driven by the first-order Markov state $M_t$. We infer that it is linear in the current belief

$$Q(\Pi_t) = \mathbb{E} \left[ Q(M_t) \mid I_t \right] = \sum_{j=1}^{d} Q(m^j)\Pi^j_t,$$

(5.3)

where $Q(m^j)$ is the P:D ratio computed under full information. The setup is highly tractable because prices are a belief-weighted average of state-prices from the full information model.

The excess return is determined by the volatility state and investor belief:

$$r_{t+1} = \ln \frac{1 + Q(\Pi_{t+1})}{Q(\Pi_t)} + \bar{\mu}_d - r_f - \frac{\sigma^2_d(M_{t+1})}{2} + \sigma_d(M_{t+1})\varepsilon_{d,t+1}. \quad (5.4)$$

When a new state occurs, investors may learn it gradually and thus generate less extreme returns than in the full information economy. Simulating the return process with learning is now straightforward, as discussed in the Appendix.

The equilibrium impact of signal variability $\sigma_\delta$ is conveniently analyzed from (5.3) for fixed values of the other structural parameters. The P:D ratio is the filtered version of its full information counterpart, which implies equality of the means: $\mathbb{E}Q(\Pi_t) = \mathbb{E}Q(M_t)$. Information quality thus has essentially no effect on the equity premium. The variance satisfies the orthogonality condition: $\text{Var}[Q(M_t)] = \text{Var}[Q(\Pi_t)] + \mathbb{E}\{[Q(\Pi_t) - Q(M_t)]^2\}$. This equation is the analogue of the variance bounds considered by Leroy and Porter (1981) and Shiller (1981). In our framework, we expect the difference in variances to be small: The variance of P:D is dominated by changes in the most persistent components. Since learning about these changes is a rare and transitory phenomenon, the difference $Q(\Pi_t) - Q(M_t)$ is likely to be modest most of the time. This suggests that the variances of P:D and returns are relatively insensitive to information quality, and we confirm this logic numerically in the next Section.

\footnote{In a representative agent economy with Epstein-Zin-Weil utility, the P:D ratio is linear in beliefs if: (1) dividend growth is driven by a Markov state; and (2) consumption growth is a separate IID process.}
The linearity property (5.3) implies that our model does not contain the “uncertainty channel” that has been previously considered in the learning literature (e.g. Veronesi, 1999; Lettau, Ludvigson and Wachter, 2004). In these models, signals are informative about both future dividend news and future marginal rates of substitution, which generates a higher sensitivity of returns about bad news in good times than about good news in bad times. Our model illustrates that even in the absence of such effects, learning about stochastic volatility can be a powerful source of endogenous skewness.

The equilibrium effect of information quality is investigated by Veronesi (2000), who examines learning about growth rates. We instead consider the impact of signal precision when investors learn about multifrequency dividend news volatility.\(^\text{16}\)

5.2. Learning Model Results

Despite the simplicity of the pricing and updating rules, econometric inference is computationally expensive in our imperfect information equilibrium. The state consists of the volatility vector \(M_{t+1}\) and the investor belief \(\Pi_{t+1}\). Since the econometrician observes only excess returns, evaluating the likelihood of the data would require integrating over the conditional distribution of the state \((\Pi_t, M_t)\). When \(k = 8\), this would entail estimating a distribution defined on \(\mathbb{R}^{256} \times \{m_1, ..., m_{256}\}\).

We instead use a simulation-based approach, and focus on the two base specifications with \(k = 8\) frequencies considered in Section 4. Specifically, we assign the daily values \(\bar{\mu}_d - r_f = 0.5\) bp, \(\bar{\sigma}_d = 0.70\)%, and set the parameters \((m_0, \gamma_k, b)\) to the full-information ML estimates reported in Table 2. Consistent with the empirical estimates and calibration in Bansal and Yaron (2004), we choose \(\rho_{c,d} = 0.6\). Unreported robustness checks show that the learning results are not highly sensitive to the choice of \(\rho_{c,d}\) over a wide range.

To evaluate the impact of information quality, we consider a set of signal volatilities \(\sigma_d \in \{0, 0.1, 1, 1.25, 2, 3, 4, 5, 10, 15, 20\}\). For each value, we simulate a single long sample of excess returns, and calculate the first four moments of returns as well as the feedback, using the same set of random draws. We report a subset of the results in Table 5. The postwar and full-sample parameters produce qualitatively similar moments; for simplicity, we focus our discussion on the postwar period. In Panel B, the average mean return is equal to 1.93 bp per day for all values of the signal precision. The simulated means are close to the average daily equity premium in the postwar period, and the constancy across simulations is consistent with the observation that the average P:D ratio is independent of the signal precision. The standard deviation is likewise

\(^{16}\)Veronesi anticipates an extension along the lines we pursue: “There are other types of information that are certainly relevant and that are also worth investigating. These may include information about future volatilities [...] for example. The effect of information quality on these variables may have different implications on stock returns than the one discussed here.” We verify this prediction.
nearly invariant to information quality, and takes a daily value of about 0.80% for each simulation, close to the empirical value in the postwar period. Feedback is accordingly nearly constant across the different simulations, and takes a value of about 29%. Thus, the degree of volatility feedback is robust across different learning environments.

We do, however, find large and systematic differences in the degree of skewness and kurtosis as signal precision varies. Skewness is close to zero at about −0.05 when $\sigma_\delta = 0$, falling to −0.44 when $\sigma_\delta = 0.5$, and to −1.06 when $\sigma_\delta = 2$. Returns become more negatively skewed as investor information becomes less precise. Kurtosis takes its highest value of about 83 when investor information is perfect. With a value of $\sigma_\delta = 0.5$ kurtosis drops to 24 and when $\sigma_\delta = 2$ kurtosis falls to 11. We thus infer a tradeoff between skewness and kurtosis. With perfect information kurtosis is large but skewness is close to zero. As the quality of investor information deteriorates, returns become more negatively skewed and kurtosis falls as well. Figure 5 depicts the tradeoff between skewness and kurtosis for $\sigma_\delta \leq 5$. Intermediate information qualities approximately in the range $\sigma_\delta \in [0.5, 1.0]$ seem most consistent with the kurtosis and negative skewness observed in the data.

To understand these results, consider the role played by dividend growth in the investor updating process. When information is perfect, dividend growth plays no role in determining investor beliefs about the volatility state. Regardless of whether volatility state variables increase, decrease, or stay the same, investors find out immediately and fully incorporate into price the impact of any changes. The speed of learning is independent of the direction of the volatility change, and returns are approximately symmetric. Kurtosis is high and skewness close to zero.

At the other extreme, when $\sigma_\delta$ is arbitrarily large the corresponding signals are not useful. Investors then rely on dividend news to infer the latent state. If volatility increases, investors may get a single extreme observation that is implausible under their existing beliefs. In this case beliefs quickly revise upward. On the other hand, a volatility decrease (good news) can only be revealed slowly. This is because investors learn about low volatility by observing dividend growth close to its mean, but this is a relatively likely outcome regardless of the volatility level. Thus, bad news about increased volatility can be incorporated into price quickly, while good news about low volatility trickles out slowly. This asymmetry explains why skewness increases and kurtosis falls as information quality about the volatility state deteriorates.

To further illustrate the effect of information quality, Figure 6 displays four simulations with length $T = 20,000$ of the learning economy with different signal precisions. Consecutively from top to bottom, $\sigma_\delta = 0$ corresponds to full information, $\sigma_\delta = 0.5$ and $\sigma_\delta = 1.0$ give two intermediate values, and $\sigma_\delta = 20$ corresponds to nearly uninformative signals. All simulations use identical sets of random draws to facilitate comparison. With perfect information, large and symmetric feedback gives substantial outliers of
both signs. As information quality decreases, gradual learning causes feedback to be spread out across multiple days, and fewer extreme returns occur. The attenuation is stronger for positive returns, and skewness thus becomes more pronounced with $\sigma_\delta$. When $\sigma_\delta = 20$, this effect is so extreme that no large positive returns occur in the simulation. The intermediate cases where $\sigma_\delta = 0.5$ and $\sigma_\delta = 1.0$ appear most consistent with daily stock returns.

6. Robustness Checks, Preference Implications, and Extension

This section discusses the role of the preference parameters in the previous empirical results. We also examine the robustness of volatility feedback to alternative specifications of consumption and dividend drift.

The preference parameters are implicitly chosen to match the average return on the bond and the stock, as discussed in Section 3.2. Specifically, the calibration of the P:D ratio implies a unique value for $\alpha \sigma_{c,d} \equiv \alpha \sigma_c \bar{\sigma}_d \rho_{c,d}$, which is equal to 2.8 basis points per day in the base postwar example reported in Table 2. As in the learning section, we choose $\rho_{c,d} = 0.6$ to produce a correlation between dividends and consumption in the range considered by Campbell (2003) and Bansal and Yaron (2004). Also following Bansal and Yaron, we calibrate aggregate consumption to US values, and use $g_c = 1.8\%$ and $\sigma_c = 2.93\%$ per year. The corresponding value of risk aversion is $\alpha \approx 35$. If the EIS is set equal to $\psi = 1$, the discount rate $\delta = 97.8\%$ per year then matches the interest rate $r_f = 1\%$.

The results in Section 4 thus imply reasonable levels of the EIS and subjective discount rate, but large relative risk aversion. Previous calibrations in the literature have used $\alpha$ in this range (e.g., Lettau, Ludvigson and Wachter use a value of 40). Nonetheless, we would like to better understand the importance of risk aversion to volatility feedback in our framework. The loglinear approximation (Calvet and Fisher, 2005) shows that $\alpha$ controls both the magnitude of the equity premium and the price impact of volatility changes. In order to achieve a reasonable equity premium using a lower risk aversion, we therefore need an additional source of risk in stock returns.

Small but persistent variations in the drift and volatility of consumption have recently been proposed as a solution to the equity premium puzzle (Bansal and Yaron, 2004). Empirical support for this hypothesis is provided by Bansal and Lundblad (2002), Bansal, Khatchatrian, and Yaron (2005) and Lettau, Ludvigson and Wachter (2004). Bansal and Yaron consider that the drift and volatility of consumption are driven by an autoregressive state with a half-life of about two and half years. Lettau, Ludvigson and Wachter estimate a four-state regime-switching model in which the good state has a duration of over 30 years.

This earlier research motivates the following extension of our asset-pricing model.
Consumption growth exhibits regime shifts in drift and volatility:
\[ c_t - c_{t-1} = g_c(M'_t) + \sigma_c(M'_t)\varepsilon_{c,t}, \]
where \( \{\varepsilon_{c,t}\} \) is IID \( \mathcal{N}(0,1) \), and \( M'_t \in \mathbb{R}^{\tilde{\ell}}_+ \) is a multifrequency state vector with \( \tilde{\ell} \) components. The components of the state \( M'_t \) each take the values \( m_{c0} > 1 \) and \( 2 - m_{c0} \) with equal probability. Drift and volatility are specified by
\[
 g_c(M'_t) \equiv \tilde{g}_c - \lambda_c \sum_{k=1}^{\tilde{\ell}} (M'_{k,t} - 1)
\]
\[
 \sigma_c(M'_t) \equiv \sigma_c(M'_{1,t}...M'_{\tilde{\ell},t})^{1/2}.
\]
Consumption volatility is the product of the components \( M'_{k,t} \), as in the dividend news process. We define the drift, on the other hand, as the sum of the state components, which permits a symmetric distribution around \( \tilde{g}_c \). Similarly, the dividend growth process (2.2) exhibits regime-switches in drift
\[
 \mu_d(M_t) = \tilde{\mu}_d - \lambda_d \sum_{k=1}^{\tilde{k}} (M_{k,t} - 1),
\]
as well as the usual MSM volatility (3.1).

The extended specification allows us to capture the variations in macroeconomic risk that have been documented at various frequencies in the literature. When \( \lambda_c > 0 \), a high component \( M'_{k,t} \) implies both a low drift and a higher volatility, which is consistent with empirical evidence on business cycles. We assume \( \tilde{\ell} \leq \tilde{k} \), consistent with the idea of consumption smoothing at short horizons. For instance, consumption may be affected by business cycles, technology and demographic shocks, but not by shorter-lived shocks that can affect dividend news. For simplicity, we assume that the consumption and volatility components are perfectly correlated: \( M'_{k,t} = m_{c0} \) if and only if \( M_{k,t} = m_0 \) for every \( k \). Asset prices are easily derived, as shown in the Appendix.

We calibrate the model using statistics reported by Bansal and Yaron (2004). We set aggregate consumption to \( g_c = 1.80\% \) per year and \( \sigma_c = 2.89\% \) per year, and dividends satisfy \( \tilde{\mu}_d = g_c, \tilde{\sigma}_d = 0.80\% \) per day. There are \( \tilde{\ell} = 4 \) consumption components and \( \tilde{k} = 5 \) dividend components. We choose the dividend volatility parameter \( m_0 = 1.50 \) close to the values estimated in Table 2, and set \( m_{c0} = 1.40 \). The drift parameter \( \lambda_c \) satisfies \( \lambda_c(m_{c0} - 1) = 0.30\% \) on an annualized basis, implying that the state-dependent consumption drift varies between 0.6% and 3% per year. Similarly, the dividend drift switches are specified by \( \lambda_d(m_0 - 1) = 0.5\% \), implying state dependent drifts varying between -0.7% and 4.3% per year. We set the correlation \( \rho_{c,d} = 0.64 \), and frequencies are specified by \( b = 2.4 \), and \( 1/\gamma_1 = 20 \) years. The durations of the consumption
components thus range between 1.44 and 20 years, while the shortest dividend duration is 0.6 years. The coefficient of relative risk aversion is \( \alpha = 10 \), and the other preference parameters are \( \psi = 1.5 \) and \( \delta = 0.993 \).

The dynamics of consumption in this calibration are consistent with current literature and existing evidence. The standard deviation of consumption growth is almost identical to the value used by Bansal and Yaron, and consumption growth autocorrelations are 0.032 at a one-year horizon, 0.015 over five years, and 0.009 over ten years. These values would be hard to distinguish from white noise, and are in fact much lower than the autocorrelations in Bansal and Yaron. Similarly, the variance ratios do not exceed 1.21 over a 10 year horizon, again lower than the values in Bansal and Yaron. The consumption specification thus appears consistent with earlier empirical evidence.

We first shut down the stochastic volatility of dividends: \( \sigma_d(M_t) \equiv \bar{\sigma}_d \). This yields a risk premium of 2.16%, average P:D ratio equal to 46.7, and an average risk-free rate equal to 1.02% per year. The stock return has variance 4.2% higher than the dividend variance, i.e., \( FB = 4.2\% \).

We now turn on the stochastic volatility of dividends. The consumption process, stochastic discount factor and interest rate regimes are unchanged. The equity premium on the stock increases to 3.29% and the P:D ratio falls to 31.1. The P:D regimes vary between 22.9 and 37.7. The feedback increases to 25.9%, which is more than 6 times larger than when dividend volatility was constant. The marginal contribution of dividend multifrequency volatility to return volatility is thus 25.9% − 4.2% = 21.7%, which is comparable to the feedback estimates obtained with IID consumption and high risk aversion.

By incorporating long-run risks in consumption, we can thus use a lower risk aversion (\( \alpha = 10 \)) to match the equity premium and still generate a substantial contribution of dividend volatility feedback. The extension also offers a pure regime-switching formulation of long-run risks in a multifrequency environment, opening new directions for future research.

7. Conclusion

This paper develops a tractable asset-pricing framework for economies with multifrequency shocks to fundamentals. We focus on a dividend news specification with constant mean, multifrequency stochastic volatility, and a conditionally Gaussian noise. The structural equilibrium with three free parameters accounts for endogenous skewness, thick tails, time-varying volatility and sizeable feedback in over eighty years of daily stock returns.

Two economic mechanisms play important roles. First, endogenous volatility feedback amplifies dividend variance by 20 to 40% in favored specifications, or 10 to 40
times the amount in previous literature (e.g., Campbell and Hentschel, 1992). Feedback from persistent components helps to capture extreme returns, while higher-frequency variations match day-to-day volatility movements. Second, investor learning generates substantial endogenous skewness. Building on Veronesi (2000), we consider investor signals about the volatility state, and show that information quality creates a tradeoff between skewness and kurtosis. Intermediate information environments best match the data.

The paper illustrates that a multifrequency approach helps to connect low-frequency literature in macro-finance and learning with higher frequency financial econometrics. Convergence of these areas follows from bringing multifrequency shocks into pure regime-switching economies, which traditionally offer three major benefits: 1) asset pricing is straightforward in a Markov chain setup; 2) the econometrics of regime-switching, based on a simple filtering theory, is well-understood; and 3) learning is easily incorporated by using similar filtering techniques. The multifrequency approach expands the practical range of equilibrium regime-switching economies from a few states to several hundred, and from lower frequencies to daily returns.

We develop an extension based on joint modelling of multifrequency regime-switches in consumption and dividends. This generates large feedback and a reasonable equity premium with moderate values of relative risk aversion. We anticipate that this framework offers considerable potential for further development, particularly in modelling the impact of long-run risks on high frequency financial data.
8. Appendix A. Full-Information Economies

8.1. Stochastic Discount Factor

As shown by Epstein and Zin (1989), a utility-maximizing agent with budget constraint $W_{t+1} = (W_t - C_t)(1 + R_{t+1})$ has stochastic discount factor

$$ SDF_{t+1} = \left[ \delta \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \right]^\theta \left[ \frac{1}{1 + R_{t+1}} \right]^{1-\theta}, $$

where $R_{t+1}$ is the simple net return on the optimal portfolio.

In our setup, the representative agent can be viewed as holding a long-lived claim on the aggregate consumption stream $\{C_t\}_{t=0}^\infty$. The tree has price $P_c C_t$, and yields the return $1 + R_{c,t+1} = \left( 1 + \frac{1}{P_c} \right) C_{t+1}/C_t$. The stochastic discount factor is thus

$$ SDF_{t+1} = \delta^\theta (1 + 1/P_c)^{\theta - 1} \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha}. $$

The condition $E_t[SDF_{t+1}(1+R_{c,t+1})] = 1$ implies that $\delta^\theta (1+1/P_c)^{\theta} E[(C_{t+1}/C_t)^{1-\alpha}] = 1$ or equivalently

$$ 1 + 1/P_c = \delta^{-1} \{E[(C_{t+1}/C_t)^{1-\alpha}]\}^{-\frac{1}{\theta}}. $$

We conclude that equation (2.3) holds.

8.2. Bayesian Updating and Closed-Form Likelihood

We derive in Calvet and Fisher (2005) the filtered probabilities $\hat{\Pi}_t = (\hat{\Pi}_j^t)_{1 \leq j \leq d}$ and the corresponding likelihood.

**Proposition 1 (Filtered Probability and Likelihood Function).** The econometrician’s conditional probabilities are computed recursively using Bayes’ rule:

$$ \hat{\Pi}_{t+1} = \frac{\hat{\Pi}_t [A \ast F(r_{t+1})]}{\hat{\Pi}_t [A \ast F(r_{t+1})] 1'}, \quad (8.1) $$

where $\ast$ denotes element-by-element multiplication, $A = (a_{i,j})_{1 \leq i,j \leq d}$ is the transition matrix of the Markov state: $a_{i,j} = P(M_{t+1} = m^j | M_t = m^i)$, $F(r)$ is the matrix with elements $F_{i,j}(r) \equiv f_{r_{t+1}}(r|M_t = m^i, M_{t+1} = m^j)$, and $1 = (1,..,1)' \in \mathbb{R}^d$. The log-likelihood of the return process satisfies

$$ \ln L(r_1,\ldots,r_T) = \sum_{t=1}^T \ln \left\{ \hat{\Pi}_{t-1} [A \ast F(r_t)] 1' \right\}. \quad (8.2) $$

26
We similarly establish the following result.

**Proposition 2 (Smoothed Probabilities).** The econometrician’s smoothed probabilities satisfy the backward recursion

\[
\hat{\Psi}_t^i = \hat{\Pi}_t^i \sum_{j=1}^{d} a_{ij} \hat{\Psi}_{t+1}^j \left[ \frac{F_{i,j}(r_{t+1})}{f_{r_{t+1}}(r_{t+1}|I_t^0)} \right],
\]

(8.3)

\(i \in \{1, \ldots, d\}\), and the final condition \(\hat{\Psi}_T = \hat{\Pi}_T\).

### 8.3. Ex Post Decomposition

We condition the return equation (3.3) with respect to the econometrician’s information set \(I_T^0\):

\[
\hat{\Psi}_t = \hat{\Pi}_t \sum_{j=1}^{d} a_{ij} \hat{\Psi}_{t+1}^j \left[ \frac{F_{i,j}(r_{t+1})}{f_{r_{t+1}}(r_{t+1}|I_t^0)} \right],
\]

(8.3)

\(i \in \{1, \ldots, d\}\), and the final condition \(\hat{\Psi}_T = \hat{\Pi}_T\).

### 8.4. The Campbell-Hentschel Model

The CH specification is based on a dividend news that follows a QGARCH(1,2) process (Engle, 1990; Sentana, 1995). Excess returns follow

\[
r_{t+1} = \hat{\mu}_d - r_f + \mathbb{E} \left[ \ln \left( \frac{1 + Q(M_{t+1})}{Q(M_t)} \right) - \frac{\sigma_d(M_{t+1})^2}{2} \left| I_T^0 \right\} + \hat{\epsilon}_{d,t+1}. \]

The definition of smoothed probabilities implies

\[
r_{t+1} = \hat{\mu}_d - r_f + \mathbb{E}_{\hat{\Psi}_t} \left[ \ln(1 + Q(M_{t+1})) - \sigma_d(M_{t+1})^2/2 \right] - \mathbb{E}_{\hat{\Psi}_t} \ln Q(M_t) + \hat{\epsilon}_{d,t+1}. \]

Since \(\mathbb{E}_{\hat{\Psi}_t} r_t = \hat{\mu}_d - r_f + \mathbb{E}_{\hat{\Psi}_t} \left( \ln(1 + Q(M_{t+1})) - \sigma_d(M_{t+1})^2/2 - \ln Q(M_t) \right)\), we conclude that (4.1) holds.
9. Appendix B: Learning Economies

Consider the volatility state and investor probability distribution \((M_t, \Pi_t)\) at the end of period \(t\). The state of the economy in the following period is computed in three steps:

1. **Latent state of nature.** We draw the volatility state \(M_{t+1}\) given \(M_t\). We also sample \(\tilde{k} + 2\) independent standard normals \((z_{1,t+1}; \ldots; z_{\tilde{k},t+1}; \epsilon_{d,t+1}; \eta_{c,t+1})\). The Gaussian consumption noise is \(\epsilon_{c,t+1} = \rho_{c,d} \epsilon_{d,t+1} + \sqrt{1 - \rho_{c,d}^2} \eta_{c,t+1}\). We then compute the consumption, dividend and signal in period \(t + 1\).

2. **Investor belief.** The investor observes \((\delta_{t+1}, c_{t+1} - c_t, d_{t+1} - d_t)\). She then computes her new probability distribution \(\Pi_{t+1}\) over volatility states with Bayes’ rule:

\[
\Pi_{t+1}^j \propto f(\delta_{t+1}, c_{t+1} - c_t, d_{t+1} - d_t | M_{t+1} = m^j) \sum_{i=1}^{d} a_{i,j} \Pi_{t}^i. \quad (9.1)
\]

3. **Stock Return.** We can then compute the corresponding excess return using \((5.4)\).

10. Appendix C : Consumption Switches

**Tree.** In the presence of consumption switches, the tree has price \(P_c(M_t^i)C_t\) and the stochastic discount factor is given by

\[
SDF_{t+1} = \delta^\theta \left[ \frac{1 + P_c(M_{t+1}^i)}{P_c(M_t^i)} \right]^{\theta-1} \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha}.
\]

We index the consumption states by \(i = 1, \ldots, N = 2^\tilde{k}\). Let \(\pi_{i,j}\) denote the transition probability from state \(i\) to state \(j\). The price:consumption ratio satisfies the fixed-point equation:

\[
P_c(i) = \delta \left( \sum_{j=1}^{N} \pi_{i,j} [1 + P_c(j)]^\theta e^{(1-\alpha)g_c(j) + \sigma_c^2(j)(1-\alpha)^2/2} \right)^{1/\theta}. \quad (10.1)
\]

The interest rate \(r_f = -\ln \mathbb{E}_t(SDF_{t+1})\) is then given by

\[
r_f(i) = -\theta \ln \delta - \ln \left\{ \sum_{j} \pi_{i,j} \left[ \frac{1 + P_c(j)}{P_c(i)} \right]^{\theta-1} e^{-\alpha g_c(j) + \sigma^2_c(j)/2} \right\}.
\]

**Stock.** The P:D ratio of the stock satisfies the fixed point equation:

\[
Q(M_t) = \delta^\theta \mathbb{E}_t \left\{ \left[ \frac{1 + P_c(M_{t+1}^i)}{P_c(M_t^i)} \right]^{\theta-1} e^{\phi(M_{t+1}, M_{t+1}^i)} [1 + Q(M_{t+1})] \right\}. \quad (10.2)
\]

where \(\phi(M_{t+1}, M_{t+1}^i) = \mu_d(M_{t+1}) - \alpha g_c(M_{t+1}^i) + \alpha^2 \sigma^2_c(M_{t+1}^i)/2 - \alpha \sigma_c(M_{t+1}^i) \sigma_d(M_{t+1}) \rho_{c,d}\).
References


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**A. Full Sample: 1926-2003**

|        | Mean (%) | 0.023 | 0.044 | -0.004 | 0.021 | 0.030 |
|        | Standard deviation (%) | 0.86  | 0.67  | 0.77  | 0.93  | 1.02  |
|        | Skewness | -1.05 | -0.70 | 0.13  | -2.83 | -0.21 |
|        | Kurtosis | 26.5  | 13.3  | 5.76  | 59.8  | 7.18  |

**B. Postwar: 1952-2003**

Notes: This table reports statistics of the first four moments of daily excess returns. For each panel, we report the statistics over the entire period, and over four evenly spaced subsamples. There is considerable variability in all four moments across subsamples.
### TABLE 2. – COMBINED CALIBRATION/ESTIMATION

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**A. Full Sample**

**B. Postwar**

**Notes:** This table shows parameter estimates for the full-information regime-switching model for a number of volatility components $k$ ranging from one to eight. The table holds constant the calibrated dividend volatility $\sigma_d = 0.7\%$ per day (about 11% per year), excess dividend growth $\mu_d - \tau_f = 0.5$ bp per day (about 1.2% per year), and an annual average P:D ratio of $Q = 25$. For each value of $k$, the MSM volatility parameters $m_0$, $\gamma_k$, and $b$ are estimated on daily data by maximum likelihood. The optimized value of the likelihood function is given by $\ln L$. The constraint on average P:D identifies the product $\alpha\sigma_{sd} \equiv \alpha\sigma_d \sigma_{c,d}$. The table reports implied statistics for the first four moments of daily returns, the duration of the lowest-frequency shock in years (LFY), and the volatility feedback (FB). Where a variable depends on time scale or units, it is noted in parentheses under the variable description using the notation “d” for day and “a” for annual. Asymptotic standard errors for the estimated parameters are reported in parentheses beneath each reported value, conditional on the values of the calibrated parameters.
### TABLE 3. – COMPARISON WITH CH

#### A. Campbell-Hentschel Model Parameter Estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Full Sample</th>
<th>(0.78)</th>
<th>Postwar</th>
<th>(0.74)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega \times 10^7$</td>
<td>1.87</td>
<td>(0.140)</td>
<td>0.53</td>
<td>(0.145)</td>
</tr>
<tr>
<td>$\alpha_1$</td>
<td>0.140</td>
<td>(0.140)</td>
<td>0.68</td>
<td>(0.088)</td>
</tr>
<tr>
<td>$\alpha_2$</td>
<td>-0.073</td>
<td>(0.01)</td>
<td>-0.088</td>
<td>(0.01)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.925</td>
<td>(0.004)</td>
<td>0.934</td>
<td>(0.005)</td>
</tr>
<tr>
<td>$b \times 10^5$</td>
<td>3.05</td>
<td>(0.18)</td>
<td>3.04</td>
<td>(0.20)</td>
</tr>
<tr>
<td>$\mu \times 10^4$</td>
<td>3.60</td>
<td>(0.53)</td>
<td>3.47</td>
<td>(0.57)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.14</td>
<td>(0.03)</td>
<td>0.47</td>
<td>(0.09)</td>
</tr>
</tbody>
</table>

#### B. Likelihood Comparison

<table>
<thead>
<tr>
<th></th>
<th>BIC $p$-value</th>
<th>Free Parameters</th>
<th>ln $L$</th>
<th>BIC</th>
<th>Vuong (1989)</th>
<th>HAC Adj</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Full Sample</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSM</td>
<td>3</td>
<td>70355.7</td>
<td>-6.7749</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>QGARCH</td>
<td>7</td>
<td>69920.7</td>
<td>-6.7311</td>
<td>&lt; 0.001</td>
<td>0.007</td>
<td></td>
</tr>
<tr>
<td><strong>Postwar</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>MSM</td>
<td>3</td>
<td>46241.6</td>
<td>-7.0527</td>
<td>0.007</td>
<td>0.073</td>
<td></td>
</tr>
<tr>
<td>QGARCH</td>
<td>7</td>
<td>46057.3</td>
<td>-7.0218</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Notes:** Panel A shows parameter estimates from the CH volatility feedback model. Panel B gives a comparison of the in-sample fit versus the multifrequency regime-switching economy. The Bayesian Information Criterion is given by $BIC = T^{-1}(−2 \ln L + NP \ln T)$. The last two columns in Panel B give $p$-values from a test that the QGARCH dividend specification dominates the MSM equilibrium by the BIC criterion. The first value uses the Vuong (1989) methodology, and the second value adjusts the test for heteroskedasticity and autocorrelation. A low $p$-value indicates that the CH specification would be rejected in favor of the MSM equilibrium.
### TABLE 4. – ALTERNATIVE CALIBRATIONS

<table>
<thead>
<tr>
<th>Calibrated Parameters</th>
<th>Estimated Parameters</th>
<th>Return Moments</th>
<th>AEP</th>
<th>ασ_c,d</th>
<th>LFY</th>
<th>FB</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ_d</td>
<td>μ_d − r_f</td>
<td>P:D</td>
<td>m_0</td>
<td>γ_k</td>
<td>b</td>
<td>ln L</td>
</tr>
<tr>
<td>0.70</td>
<td>0.005</td>
<td>25.0</td>
<td>1.435</td>
<td>0.058</td>
<td>2.19</td>
<td>70355.7</td>
</tr>
<tr>
<td>0.70</td>
<td>0.005</td>
<td>30.0</td>
<td>1.435</td>
<td>0.057</td>
<td>2.24</td>
<td>70347.0</td>
</tr>
<tr>
<td>0.70</td>
<td>0.010</td>
<td>25.0</td>
<td>1.430</td>
<td>0.061</td>
<td>2.12</td>
<td>70365.3</td>
</tr>
<tr>
<td>0.70</td>
<td>0.010</td>
<td>30.0</td>
<td>1.432</td>
<td>0.059</td>
<td>2.14</td>
<td>70359.5</td>
</tr>
<tr>
<td>0.80</td>
<td>0.005</td>
<td>25.0</td>
<td>1.422</td>
<td>0.059</td>
<td>2.24</td>
<td>70378.6</td>
</tr>
<tr>
<td>0.60</td>
<td>0.005</td>
<td>25.0</td>
<td>1.475</td>
<td>0.066</td>
<td>2.73</td>
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<tr>
<td>0.50</td>
<td>0.005</td>
<td>25.0</td>
<td>1.514</td>
<td>0.085</td>
<td>3.23</td>
<td>70261.8</td>
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<td>0.70</td>
<td>0.005</td>
<td>30.0</td>
<td>1.435</td>
<td>0.057</td>
<td>2.22</td>
<td>70347.0</td>
</tr>
<tr>
<td>0.70</td>
<td>0.010</td>
<td>25.0</td>
<td>1.430</td>
<td>0.061</td>
<td>2.12</td>
<td>70365.3</td>
</tr>
<tr>
<td>0.70</td>
<td>0.010</td>
<td>30.0</td>
<td>1.432</td>
<td>0.059</td>
<td>2.14</td>
<td>70359.5</td>
</tr>
<tr>
<td>0.80</td>
<td>0.005</td>
<td>25.0</td>
<td>1.369</td>
<td>0.047</td>
<td>2.15</td>
<td>46241.6</td>
</tr>
<tr>
<td>0.60</td>
<td>0.005</td>
<td>25.0</td>
<td>1.371</td>
<td>0.045</td>
<td>2.19</td>
<td>46232.6</td>
</tr>
<tr>
<td>0.50</td>
<td>0.005</td>
<td>25.0</td>
<td>1.365</td>
<td>0.049</td>
<td>2.07</td>
<td>46249.0</td>
</tr>
<tr>
<td>0.70</td>
<td>0.005</td>
<td>30.0</td>
<td>1.367</td>
<td>0.047</td>
<td>2.10</td>
<td>46245.1</td>
</tr>
<tr>
<td>0.80</td>
<td>0.005</td>
<td>25.0</td>
<td>1.323</td>
<td>0.064</td>
<td>2.07</td>
<td>46237.3</td>
</tr>
<tr>
<td>0.60</td>
<td>0.005</td>
<td>25.0</td>
<td>1.372</td>
<td>0.057</td>
<td>2.25</td>
<td>46314.1</td>
</tr>
<tr>
<td>0.50</td>
<td>0.005</td>
<td>25.0</td>
<td>1.405</td>
<td>0.048</td>
<td>2.49</td>
<td>46217.0</td>
</tr>
</tbody>
</table>

**Notes:** This table shows parameter estimates conditional on alternative calibrations of the structural parameters in the MSM equilibrium. All estimated economies use k = 8 components. The first group of four rows in each panel holds constant average dividend volatility σ_d = 0.7% per day (about 11% per year), and considers combinations of excess dividend growth μ_d − r_f ∈ {0.5, 1.0} bp per day (about 1.2% or 2.4% per year) and annual average P:D ratio Q ∈ {25, 30}. For each combination, the MSM volatility parameters m_0, γ_k, and b are re-estimated on daily data by maximum likelihood. The optimized value of the likelihood function is given by ln L. The table reports implied statistics for the first four moments of daily returns. Excess dividend growth and average P:D determine the annual equity premium (AEP), and the constraint on average P:D identifies the product ασ_c,d ≡ ασ_dγ_kρ_c,d. Feedback is denoted FB, and LFY gives the duration of the lowest-frequency shock in years. Where a variable depends on time scale or units, it is noted in parentheses under the variable description using the notation “d” for day and “a” for annual. Asymptotic standard errors for the estimated parameters are reported in parentheses beneath each reported value, conditional on the values of the calibrated parameters. In parentheses beneath the implied moments, simulated p-values are reported for the test that the corresponding moment of the data exceeds the model moment under the null of correct specification. The second grouping of three rows in each panel holds μ_d − r_f and average P:D constant at their original values, and considers alternative values of dividend growth volatility σ_d ∈ {0.5, 0.6, 0.8} percent per day, or approximately 7.7, 9.3, and 12.4 percent annually.
TABLE 5. – MOMENTS OF THE LEARNING MODEL

<table>
<thead>
<tr>
<th>Signal Standard Deviation $\sigma_d$</th>
<th>0</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E[r_t]$</td>
<td>0.019</td>
<td>0.019</td>
<td>0.019</td>
<td>0.019</td>
<td>0.019</td>
<td>0.019</td>
</tr>
<tr>
<td>$Var[r_t]^{1/2}$</td>
<td>0.825</td>
<td>0.825</td>
<td>0.824</td>
<td>0.825</td>
<td>0.825</td>
<td>0.824</td>
</tr>
<tr>
<td>$Skew[r_t]$</td>
<td>-0.069</td>
<td>-0.127</td>
<td>-0.509</td>
<td>-0.926</td>
<td>-1.076</td>
<td>-1.181</td>
</tr>
<tr>
<td>$Kurt[r_t]$</td>
<td>133.2</td>
<td>99.2</td>
<td>42.6</td>
<td>22.9</td>
<td>17.5</td>
<td>15.5</td>
</tr>
<tr>
<td>Feedback</td>
<td>38.9</td>
<td>38.9</td>
<td>38.7</td>
<td>38.8</td>
<td>38.8</td>
<td>38.6</td>
</tr>
</tbody>
</table>

A. Full Sample: 1926-2003

| $E[r_t]$                            | 0.019 | 0.019 | 0.019 | 0.019 | 0.019 | 0.019 |
| $Var[r_t]^{1/2}$                    | 0.796 | 0.796 | 0.796 | 0.796 | 0.796 | 0.796 |
| $Skew[r_t]$                         | -0.054 | -0.097 | -0.441 | -0.804 | -0.958 | -1.06 |
| $Kurt[r_t]$                         | 83.4 | 54.1 | 23.5 | 13.8 | 11.5 | 10.7 |
| Feedback                            | 29.4 | 29.4 | 29.4 | 29.1 | 29.2 | 29.2 |

B. Postwar: 1952-2003

Notes: This table shows the effect of signal precision on different moments of daily excess returns in the learning model. For each panel, the base parameters $m_0$, $\bar{\sigma}_d$, $\bar{\mu}_d - r_f$, $\gamma_k$, and $b$ are taken from the estimates in Table 2 for the specification with $\bar{k} = 8$ components. These values, as well as the calibrated value $\rho_{c,d} = 0.6$ are held constant across all simulations. Columns vary only by the value of the reported signal standard deviation $\sigma_d$. When $\sigma_d = 0$, information is perfect, and as $\sigma_d$ becomes larger the signal precision weakens. For each specification, we simulate a single series of $T = 10^7$ returns using the same set of random draws, and report moments of the simulated data. Mean, variance, and feedback are nearly constant across simulations. Skewness becomes more negative and kurtosis declines as information quality deteriorates.
Figure 1: Daily Excess Returns. This figure shows daily excess returns in percentage points from 1926 to 2003. The market return series splices the Schwert (1990a) data from 1926-1963 with the CRSP value weighted index from 1963-2003. The risk free rate is proxied by the return on 30 day U.S. Treasury bills.
Figure 2: Conditional Probabilities of Volatility Components. This figure shows *ex ante* and *ex post* probabilities that each volatility component takes the high value in the full information MSM equilibrium with $k = 8$ components, and parameters estimated from postwar data. The filtered probabilities $\Pi_t$ are in the left-hand column, and the smoothed probabilities $\Psi_t$ are in the right-hand column. Volatility components progress from low ($k = 1$) to high ($k = 8$) frequency from top to bottom of the figure.
Figure 3: Ex Ante Conditional Mean and Volatility. This figure shows the conditional mean and variance of daily excess returns under the full information MSM equilibrium with $k = 8$ volatility components, and parameters estimated from postwar data. Conditioning information is the *ex ante* information set of returns up to and including date $t$. 
Figure 4: Ex Post Return Decomposition. This figure shows an ex post decomposition of daily excess returns using the full information MSM equilibrium with $k = 8$ components and parameters estimated on postwar data. The decomposition uses the smoothed beliefs $\Psi_t$ obtained by using the conditioning information set of all returns. The first panel shows actual returns. The second panel shows the mean return at time $t + 1$ conditional on the beliefs $\Psi_t$. The third panel shows the estimated price variation due to volatility feedback at time $t + 1$ conditional on all of the data. The final panel is the smoothed estimate of dividend news, equal to the realized return less the second and third panels.
Figure 5: Learning Model Skewness and Kurtosis. This figure shows the effect of signal precision on skewness and kurtosis. For each curve, the base parameters $m_0, \bar{\sigma}_d, \bar{\mu}_d - \tau_f, \gamma_b, \text{ and } b$ are taken from the estimates in Table 2 for the specification with $k = 8$ components. These values, as well as $\rho_{e,d} = 0.6$ are held constant across all simulations. Each simulation is then based on a different value of the signal standard deviation $\sigma_\delta \in \{0, 0.1, \ldots, 1, 1.25, \ldots, 2, 3, 4, 5\}$. When $\sigma_\delta = 0$, information is perfect, and as $\sigma_\delta$ becomes larger the signal precision weakens. For each economy, we simulate a single long series of $T = 10^7$ returns using the same set of random draws. Each marked point on the plot represents a different simulation, progressing from $\sigma_\delta = 0$ in the top left to $\sigma_\delta = 5$ in the bottom right.
Figure 6: Simulations of the Learning Economy. This figure shows learning economy simulations with length $T = 20,000$. All simulations are based on the base parameters estimated from the full sample with $k = 8$ components: $m_0 = 1.435$, $\bar{\sigma}_d = 0.70\%$ per day, $\bar{\mu}_d - r_f = 0.5$ bp per day, $\gamma_k = 0.058$, and $b = 2.19$. All simulations also use the identical set of random draws for dividends, signal noises, and multipliers. The only variable that changes across simulations is the signal variability, $\sigma_\delta$. The top panel where $\sigma_\delta = 0$ corresponds to perfect information; the middle two panels show intermediate signal precision; and in the final panel information quality is poor. Noise in the investor signals attenuates extreme feedback realizations, but attenuation is stronger for positive than negative realizations. This generates increasingly negative skewness and reduces kurtosis as information quality deteriorates.