Attilio Meucci
Lehman Brothers, Inc., New York

personal website: www.symmys.com

Issues in Statistical Trading and Quantitative Portfolio Management *
Modeling, Estimation Risk, and Robust Allocation

* for a thorough introduction to these and related issues and for references see:
AGENDA

Quantitative recipes

Estimation vs. modeling

Classical optimization

Robust optimization

Robust Bayesian optimization
AGENDA

Quantitative recipes

- Statistical trading: fixed income PCA
- Portfolio management: funds of funds

Estimation vs. modeling

Classical optimization

Robust optimization

Robust Bayesian optimization
1. consider N series of T observations of homogeneous forward rates $X$ (T×N panel)

2. define $S = XX'$ (N×N positive definite matrix)

3. run PCA $S = E \Lambda E'$ (eigenvectors-eigenvalues-eigenvectors)
STATHISTICAL TRADING RECIPE – fixed-income PCA

1. consider N series of T observations of homogeneous forward rates
   \( X \quad \text{(TxDN panel)} \)

2. define \( S \equiv XX' \quad \text{(NxN positive definite matrix)} \)

3. run PCA \( S \equiv E \Lambda E' \quad \text{(eigenvectors-eigenvalues-eigenvectors)} \)

4. analyze the series \( y \equiv Xe(N) \) of the last factor
   • z-score: structural bands
   • “juice”: b.p. from mean
   • roll-down/slide-adjusted prospective Sharpe ratio
   • reversion timeframe
   • market events (e.g. Fed, Thursday “numbers”,…)
   • relation with other series (e.g. oil prices)

5. convert basis points to PnL/risk exposure by dv01

variations: transform series, include mean, support series (PCA-regression),…
1. Consider N series of T observations of homogeneous forward rates
   \( X \) (TxN panel)

2. Define \( S \equiv XX' \) (NxN positive definite matrix)

3. Run PCA \( S \equiv E\Lambda E' \) (eigenvectors-eigenvalues-eigenvectors)

4. Analyze the series \( y \equiv Xe^{(N)} \) of the last factor
   - z-score: structural bands
   - “juice”: b.p. from mean
   - roll-down/slide-adjusted prospective Sharpe ratio
   - reversion timeframe
   - market events (e.g. Fed, Thursday “numbers”,…)
   - relation with other series (e.g. oil prices)

5. Convert basis points to PnL/risk exposure by dv01

- **estimation** (backward-looking) and **projection/modeling** (forward-looking) overlap
  - **non-linearities** not accounted for
AGENDA

Quantitative recipes

- Statistical trading: fixed income PCA
- Portfolio management: funds of funds

Estimation vs. modeling

Classical optimization

Robust optimization

Robust Bayesian optimization
PORTFOLIO OPTIMIZATION RECIPE – fund of funds

1. consider N series of T observations of fund prices $P$ (TxN panel)

2. consider the compounded returns $C_{t,n} \equiv \ln\left(P_{t,n}\right) - \ln\left(P_{t-1,n}\right)$

3. estimate covariance (e.g. the sample non-central) $\hat{\Sigma} \equiv \frac{1}{T} \sum_{t=1}^{T} C_t C_t'$

4. define the expected values (e.g. risk-premium) $\hat{\mu} \equiv \gamma \text{ diag}\left(\hat{\Sigma}\right)$
1. Consider N series of T observations of fund prices $P$ (TxN panel)

2. Consider the compounded returns $C_{t,n} \equiv \ln(P_{t,n}) - \ln(P_{t-1,n})$

3. Estimate covariance (e.g., the sample non-central) $\hat{\Sigma} \equiv \frac{1}{T} \sum_{t=1}^{T} C_t C_t'$

4. Define the expected values (e.g., risk-premium) $\hat{\mu} \equiv \gamma \text{diag}(\hat{\Sigma})$

5. Solve mean-variance: $w^{(i)} \equiv \arg\max \left\{ w' \hat{\mu} \right\}$

   - $w \in C$
   - $w' \hat{\Sigma} w \leq \nu^{(i)}$

   investment constraints, grid of significant variances

6. Choose the most suitable combination among $w^{(i)}$ according to preferences
PORTFOLIO OPTIMIZATION RECIPE – fund of funds

1. consider N series of T observations of fund prices \( P \) (TxN panel)

2. consider the compounded returns \( C_{t,n} \equiv \ln(P_{t,n}) - \ln(P_{t-1,n}) \)

3. estimate covariance (e.g. the sample non-central) \( \hat{\Sigma} \equiv \frac{1}{T} \sum_{t=1}^{T} C_t C_t' \)

4. define the expected values (e.g. risk-premium) \( \hat{\mu} \equiv \gamma \text{ diag}(\hat{\Sigma}) \)

5. solve mean-variance: \( w^{(i)} \equiv \arg\max \left\{ w' \hat{\mu} \right\} \)

6. choose the most suitable combination among \( w^{(i)} \) according to preferences

- estimation (backward-looking) and modeling (forward-looking) overlap
- projection (investment horizon) not accounted for
- non-linearities of compounded returns not accounted for
AGENDA

Quantitative recipes

Estimation vs. modeling

Classical optimization

Robust optimization

Robust Bayesian optimization
ESTIMATION VS. MODELING – general conceptual framework

- estimation
  - time series analysis

- projection
  - investment horizon

- modeling & optimization
  - P&L

- investment decision
ESTIMATION VS. MODELING – fund of funds

- Estimation: compounded returns \[ C_t^\tau = \ln(P_t) - \ln(P_{t-\tau}) \] estimation interval

  compounded returns are more symmetric, in continuous time they can be modeled (in first approximation) as a Brownian motion
ESTIMATION VS. MODELING – fund of funds

• Estimation: compounded returns

\[ C_{t}^{\tau} \equiv \ln(P_{t}) - \ln(P_{t-\tau}) \]

compounded returns are more symmetric, in continuous time they can be modeled (in first approximation) as a Brownian motion.

• Projection to investment horizon

\[ C_{t}^{\tau} = C_{t-J\tau}^{\tau} + C_{t-(J-1)\tau}^{\tau} + \cdots + C_{t}^{\tau} \]

compounded returns can be easily projected to the investment horizon because they are additive (“accordion” expansion).
ESTIMATION VS. MODELING – fund of funds

• Estimation: compounded returns $C_t^\tau \equiv \ln(P_t) - \ln(P_{t-\tau})$

  compounded returns are more symmetric, in continuous time they can be modeled (in first approximation) as a Brownian motion

• Projection to investment horizon $C_t^\tau = C_{t-J_\tau}^\tau + C_{t-(J-1)\tau}^\tau + \cdots + C_t^\tau$

  compounded returns can be easily projected to the investment horizon because they are additive (“accordion” expansion)

• Modeling: linear returns $L_t^\tau \equiv P_t / P_{t-\tau} - 1$

  linear returns are related to portfolio quantities (P&L): $L_\Pi = w'L$

  portfolio return

  securities’ returns

  securities’ relative weights

  compounded returns are NOT related to portfolio quantities (P&L): $C_\Pi \neq w'C$
ESTIMATION VS. MODELING – fund of funds

- Estimation: compounded returns
  \[ C_t^{\tilde{\tau}} \equiv \ln(P_t) - \ln(P_{t-\tilde{\tau}}) \]

  sample/risk-premium:
  \[ \Sigma^{\tilde{\tau}} \equiv \frac{1}{T} \sum_{t=1}^{T} C_t^{\tilde{\tau}} C_t^{\tilde{\tau}}', \quad \mu^{\tilde{\tau}} \equiv \gamma \text{ diag} \left( \Sigma^{\tilde{\tau}} \right) \]

- Projection to investment horizon
  \[ C_t^{\tau} \equiv C_{t-J^{\tilde{\tau}}}^{\tilde{\tau}} + C_{t-(J-1)^{\tilde{\tau}}}^{\tilde{\tau}} + \cdots + C_t^{\tilde{\tau}} \]

  “square root rule”:
  \[ \Sigma^{\tau} \equiv \frac{\tau}{\tilde{\tau}} \Sigma^{\tilde{\tau}}, \quad \mu^{\tau} \equiv \frac{\tau}{\tilde{\tau}} \mu^{\tilde{\tau}} \]

- Modeling: linear returns
  \[ L_t^{\tau} \equiv \frac{P_t}{P_{t-\tau}} - 1 \]

  Black-Scholes assumption: (log-normal)
  \[ m_n \equiv E \{ L_{t,n}^{\tau} \} = e^{\frac{\tau}{2} \left( \mu_n^{\tilde{\tau}} + \frac{1}{2} \Sigma_{nn}^{\tilde{\tau}} \right)} \]
  \[ S_{nm} \equiv Cov \{ L_{t,n}^{\tau}, L_{t,m}^{\tau} \} = e^{\frac{\tau}{2} \left( \mu_n^{\tilde{\tau}} + \frac{1}{2} \Sigma_{nn}^{\tilde{\tau}} + \mu_m^{\tilde{\tau}} + \frac{1}{2} \Sigma_{mm}^{\tilde{\tau}} \right)} \left( e^{\frac{\tau}{2} \Sigma_{nm}^{\tilde{\tau}}} - 1 \right) \]
**ESTIMATION VS. MODELING – fund of funds**

- **Estimation:** compounded returns
  \[ C_{t}^\tau = \ln\left(P_{t}\right) - \ln\left(P_{t-\tau}\right) \]

  sample/risk-premium: \( \hat{\Sigma}^\tau \equiv \frac{1}{T} \sum_{t=1}^{T} C_{t}^\tau C_{t}^\tau \) \( \hat{\mu} \equiv \gamma \text{diag} \left( \hat{\Sigma}^\tau \right) \)

- **Projection to investment horizon**
  \[ C_{t}^\tau \equiv C_{t-J\tau}^\tau + C_{t-(J-1)\tau}^\tau + \cdots + C_{t}^\tau \]

  “square root rule”:
  \[ \hat{\Sigma}^\tau \equiv \frac{\tau}{\tilde{\tau}} \hat{\Sigma}^{\tilde{\tau}} \]
  \[ \hat{\mu} \equiv \frac{\tau}{\tilde{\tau}} \hat{\mu} \]

- **Modeling:** linear returns
  \[ L_{t}^\tau \equiv \frac{P_{t}}{P_{t-\tau}} - 1 \]

  Black-Scholes assumption: (log-normal)
  \[ m_{n} \equiv E \left\{ L_{t,n}^\tau \right\} = e^{\frac{\tau}{2}\left(\mu_{n} + \frac{1}{2}\Sigma_{nn}\right)} \]

  \[ S_{nm} \equiv \text{Cov} \left\{ L_{t,n}^\tau, L_{t,m}^\tau \right\} = e^{\frac{\tau}{2}\left(\mu_{n} + \frac{1}{2}\Sigma_{nn} + \mu_{m} + \frac{1}{2}\Sigma_{mm}\right)} \left( e^{\frac{\tau}{2}\Sigma_{nm}} - 1 \right) \]

  the mean - variance optimization can be fed with the appropriate inputs
AGENDA

Quantitative recipes

Estimation vs. modeling

Classical optimization

Robust optimization

Robust Bayesian optimization
CLASSICAL OPTIMIZATION – MV in theory …

\[ w^{(i)} \equiv \arg \max_w \{ w'm \} \]

subject to
\begin{align*}
    w & \in C \\
    w'Sw & \leq \nu^{(i)}
\end{align*}

\[ m \equiv \mathbb{E}\left\{ L^\tau_{t+\tau} \right\} \]

\[ S \equiv \text{Cov}\left\{ L^\tau_{t+\tau} \right\} \]

\( w \): relative portfolio weights

\( C \): set of investment constraints

\( \nu^{(i)} \): significant grid of target variances

\( m \): expected value of future returns
CLASSICAL OPTIMIZATION – … MV in practice

\[ \mathbf{w}^{(i)} \equiv \arg\max_w \{ \mathbf{w}' \mathbf{m} \} \]

subject to

\[ \mathbf{w} \in \mathcal{C} \]

\[ \mathbf{w}' \mathbf{S}\mathbf{w} \leq \nu^{(i)} \]

\[ \mathbf{w}^{(i)} \equiv \arg\max_w \{ \mathbf{w}' \hat{\mathbf{m}} \} \]

subject to

\[ \mathbf{w} \in \mathcal{C} \]

\[ \mathbf{w}' \hat{\mathbf{S}}\mathbf{w} \leq \nu^{(i)} \]

\( \mathbf{w} \): relative portfolio weights

(\( \mathcal{C} \)): set of investment constraints

\( \nu^{(i)} \): significant grid of target variances

\( \mathbf{m} \equiv \mathbb{E}\{ \mathbf{L}_{t+\tau}^r \} \)

\( \hat{\mathbf{m}} \): estimate of \( \mathbf{m} \)

\( \mathbf{S} \equiv \text{Cov}\{ \mathbf{L}_{t+\tau}^r \} \)

\( \hat{\mathbf{S}} \): estimate of \( \mathbf{S} \)
AGENDA

Quantitative recipes

Estimation vs. modeling

Classical optimization

Robust optimization

- Theory
  - Practice: mean-variance

Robust Bayesian optimization
The true optimal allocation is determined by a set of parameters that are estimated with some error:

$$\hat{\theta} \equiv (\hat{m}, \hat{S}) \neq \theta \equiv (m, S)$$
The true optimal allocation is determined by a set of parameters that are estimated with some error:

\[ \hat{\theta} \equiv \left( \hat{m}, \hat{S} \right) \neq \theta \equiv \left( m, S \right) \]

- The classical “optimal” allocation based on point estimates \( \hat{\theta} \equiv \left( \hat{m}, \hat{S} \right) \) is sub-optimal
- More importantly, the sub-optimality due to estimation error is large (Jobson & Korkie (1980); Best & Grauer (1991); Chopra & Ziemba (1993))
• The **point estimate** for the parameters must be replaced by an **uncertainty region** that includes the true, unknown parameters:

$$\Theta \equiv \left( \hat{m}, \hat{S} \right) \implies \hat{\Theta}$$
ROBUST OPTIMIZATION – estimation risk: solution

• The point estimate for the parameters must be replaced by an uncertainty region that includes the true, unknown parameters:

\[
\hat{\theta} \equiv (\hat{m}, \hat{S}) \quad \mapsto \quad \Theta
\]

• The allocation **optimization** must be performed over all the parameters in the uncertainty region:

\[
w^{(i)} \equiv \arg\max_{w \in C} \{\ldots\} \quad \mapsto \quad w^{(i)} \equiv \arg\max_{w \in C} \{\ldots\}
\]
AGENDA

Quantitative recipes

Estimation vs. modeling

Classical optimization

Robust optimization

- Theory
- Practice: mean-variance

Robust Bayesian optimization
ROBUST OPTIMIZATION – from the standard MV …

\[ w^{(i)} \equiv \arg\max_w \left\{ w' \hat{m} \right\} \]

subject to \[ w \in \mathcal{C} \]

\[ w' \hat{S} w \leq \nu^{(i)} \]

\( w \) : relative portfolio weights

\( \mathcal{C} \) : set of investment constraints

\( \nu^{(i)} \) : significant grid of target variances

\( \hat{m} \) : (point) estimate of \( m \)

\( \hat{S} \) : (point) estimate of \( S \)
ROBUST OPTIMIZATION – ... to a conservative MV approach

\[ w^{(i)} \equiv \arg\max_w \left\{ w'\hat{m} \right\} \]

subject to

\[ w \in C \]

\[ w'\hat{S}w \leq \nu^{(i)} \]

\[ w^{(i)} \equiv \arg\max_w \left\{ \min_{m \in \Theta_m} w' m \right\} \]

subject to

\[ w \in C \]

\[ \max_{S \in \Theta_S} \left\{ w'Sw \right\} \leq \nu^{(i)} \]

\[ w \]: relative portfolio weights

\[ C \]: set of investment constraints

\[ \nu^{(i)} \]: significant grid of target variances

\[ \hat{m} \]: (point) estimate of \( m \)

\[ \Theta_m \]: uncertainty set for \( m \)

\[ \hat{S} \]: (point) estimate of \( S \)

\[ \Theta_S \]: uncertainty set for \( S \)
ROBUST OPTIMIZATION – uncertainty regions

Trade-off for the choice of the uncertainty regions:

- Must be as large as possible, in such a way that the true, unknown parameters (most likely) are captured
- Must be as small as possible, to avoid trivial and nonsensical results
AGENDA

Quantitative recipes

Estimation vs. modeling

Classical optimization

Robust optimization

Robust Bayesian optimization
  • Theory
  • Practice: mean-variance
  • An example
The Bayesian approach to estimation of the generic market parameters $\theta \equiv (m, S)$ differs from the classical approach in two respects:

- it blends historical information from time series analysis with experience
- the outcome of the estimation process is a (posterior) distribution, instead of a number
in the Bayesian approach the expected values of the returns are a random variable
in the Bayesian approach the covariance matrix of the returns is a random variable
Robust allocations are guaranteed to perform adequately for all the markets within the given uncertainty ranges.

However…

- the uncertainty regions for the market parameters are somewhat arbitrary
- the investor's experience, or prior knowledge, is not considered
Robust allocations are guaranteed to perform adequately for all the markets within the given uncertainty ranges.

However…

- the uncertainty regions for the market parameters are somewhat arbitrary
- the investor's experience, or prior knowledge, is not considered

**Bayesian** approach to parameter estimation within the **robust** framework:

- naturally identifies a suitable **uncertainty regions** for the market parameters
- includes within a sound statistical framework the investor’s **prior experience**
The Bayesian posterior distribution defines naturally a self-adjusting uncertainty region $\hat{\Theta}^q$ for the market parameters.

This region is the location-dispersion ellipsoid defined by:

- a location parameter: the classical-equivalent estimator $\hat{\theta}_{ce}$
- a dispersion parameter: the positive symmetric scatter matrix $S_\theta$
- a radius factor $q$

\[
\hat{\Theta}^q : \left( \theta - \hat{\theta}_{ce} \right)' S_\theta^{-1} \left( \theta - \hat{\theta}_{ce} \right) \leq q^2
\]
ROBUST BAYESIAN OPTIMIZATION – Bayesian ellipsoids

Standard choices for the classical equivalent and the scatter matrix respectively:

- global picture: expected value / covariance matrix

\[ \hat{\theta}_{ce} \equiv \int \theta f_{po}(\theta) d\theta \]
\[ S_\theta \equiv \int \left( \theta - \hat{\theta}_{ce} \right) \left( \theta - \hat{\theta}_{ce} \right)^\prime f_{po}(\theta) d\theta \]

- local picture: mode / modal dispersion

\[ \hat{\theta}_{ce} \equiv \arg\max_\theta \left\{ f_{po}(\theta) \right\} \]
\[ S_\theta \equiv -\left( \frac{\partial \ln f_{po}(\theta)}{\partial \theta \partial \theta'} \bigg|_{\hat{\theta}_{ce}} \right)^{-1} \]
AGENDA

Quantitative recipes

Estimation vs. modeling

Classical optimization

Robust optimization

Robust Bayesian optimization
  • Theory
  • Practice: mean-variance
  • An example
ROBUST BAYESIAN MV – from the standard MV ...

\[ w^{(i)} \equiv \arg \max_{w \in C} \left\{ w' \hat{m} \right\} \]

subject to \[ w' \hat{S} w \leq \nu^{(i)} \]

- **\( w \)**: relative portfolio weights
- **\( C \)**: set of investment constraints
- **\( \nu^{(i)} \)**: significant grid of target variances
- **\( \hat{m} \)**: (point) estimate of **\( m \)**
- **\( \hat{S} \)**: (point) estimate of **\( S \)**
ROBUST BAYESIAN MV – … to the conservative robust MV …

\[ w^{(i)} \equiv \arg\max_{w \in \mathcal{C}} \left\{ w' \hat{m} \right\} \]

subject to \[ w' \hat{S} w \leq \nu^{(i)} \]

\[ w \] : relative portfolio weights

\[ \mathcal{C} \] : set of investment constraints

\[ \nu^{(i)} \] : significant grid of target variances

\[ \hat{m} \] : (point) estimate of \[ m \]

\[ \hat{S} \] : (point) estimate of \[ S \]

\[ \hat{m} \] : (point) estimate of \[ m \]

\[ \Theta_m \] : uncertainty set for \[ m \]

\[ \Theta_S \] : uncertainty set for \[ S \]
ROBUST BAYESIAN MV – … to the robust BMV

\[
\mathbf{w}^{(i)} \equiv \arg \max_{w \in \mathcal{C}} \{ \mathbf{w}' \hat{\mathbf{m}} \}
\]

subject to \( \mathbf{w}' \hat{\mathbf{S}} \mathbf{w} \leq \nu^{(i)} \)

\[
\mathbf{w}^{(i)}_{p,q} \equiv \arg \min_{w \in \mathcal{C}} \left\{ \min_{m \in \hat{\Theta}_q^m} \mathbf{w}' \mathbf{m} \right\}
\]

subject to \( \max_{\mathbf{S} \in \hat{\Theta}_q^s} \{ \mathbf{w}' \mathbf{S} \mathbf{w} \} \leq \nu^{(i)} \)

\( \mathbf{w} \): relative portfolio weights

\( \mathcal{C} \): set of investment constraints

\( \nu^{(i)} \): significant grid of target variances

\( \hat{\mathbf{m}} \): (point) estimate of \( \mathbf{m} \)

\( \hat{\Theta}_q^m \): Bayesian ellipsoid of radius \( q \) for \( \mathbf{m} \)

\( \hat{\mathbf{S}} \): (point) estimate of \( \mathbf{S} \)

\( \hat{\Theta}_p^s \): Bayesian ellipsoid of radius \( p \) for \( \mathbf{S} \)
ROBUST BAYESIAN MV – 3-d frontier

The robust Bayesian efficient allocations $\mathbf{w}^{(i)}_{p,q}$ represent a three-dimensional frontier parametrized by:

1. **Exposure to market risk** represented by the target variance $\mathbf{v}^{(i)}$

2. **Aversion to estimation risk** for the expected returns $\mathbf{m}$ represented by radius $q$

   ...indeed, a large ellipsoid $\Theta^q_m$ corresponds to an investor that is very worried about poor estimates of $\mathbf{m}$

3. **Aversion to estimation risk** for the returns covariance $\mathbf{\Sigma}$ represented by radius $p$

   ...indeed, a large ellipsoid $\Theta^p_S$ corresponds to an investor that is very worried about poor estimates of $\mathbf{\Sigma}$
AGENDA

Quantitative recipes

Estimation vs. modeling

Classical optimization

Robust optimization

Robust Bayesian optimization

- Theory
- Practice: mean-variance
- An example
We make the following assumptions:

- The market is composed of equity-like securities, for which the returns are independent and identically distributed across time.
- The estimation interval coincides with the investment horizon.
- The linear returns are normally distributed:

\[ L_{t+\tau}^\tau \mid m, S \sim N(m, S) \]

We model the investor’s prior as a normal-inverse-Wishart distribution:

\[ m \mid S \sim N\left(m_0, \frac{S}{T_0}\right), \quad S^{-1} \sim \mathcal{W}\left(v_0, \frac{S_0^{-1}}{v_0}\right) \]

where

- \((m_0, S_0)\): investor’s experience on \((m, S)\)
- \((T_0, v_0)\): investor’s confidence on \((m_0, S_0)\)
ROBUST BAYESIAN MV – posterior distribution of market parameters

Under the above assumptions, the posterior distribution is normal-inverse-Wishart, see e.g. Aitchison and Dunsmore (1975):

\[ m \mid S \sim N \left( m_1, \frac{S}{T_1} \right), \quad S^{-1} \sim W \left( \nu_1, \frac{S^{-1}}{\nu_1} \right) \]

where

\[
\hat{m} \equiv \frac{1}{T} \sum_{i=1}^{T} l_{i}^{\tau}
\]

\[ T_1 \equiv T_0 + T \]

\[
\hat{m}_1 \equiv \frac{1}{T_1} \left[ T_0 m_0 + T \hat{m} \right]
\]

\[
\hat{S} \equiv \frac{1}{T} \sum_{i=1}^{T} (l_{i}^{\tau} - \hat{m})(l_{i}^{\tau} - \hat{m})'
\]

\[ \nu_1 \equiv \nu_0 + T \]

\[
\hat{S}_1 \equiv \frac{1}{\nu_1} \left[ \nu_0 S_0 + T \hat{S} + \frac{(m_0 - \hat{m})(m_0 - \hat{m})'}{\frac{1}{T_0} + \frac{1}{T}} \right]
\]
The certainty equivalent and the scatter matrix for the posterior (Student t) marginal distribution of $m$ are computed in Meucci (2005):

$$m_{ce} = m_1, \quad S_m = \frac{1}{T_1} \frac{\nu_1}{\nu_1 - 2} S_1$$

The certainty equivalent and the scatter matrix for the posterior (inverse-Wishart) marginal distribution of $S$ are computed in Meucci (2005):

$$S_{ce} = \frac{\nu_1}{\nu_1 + N + 1} S_1, \quad S_S = \frac{2\nu_1^2}{(\nu_1 + N + 1)^3} \left( D_N' \left( S_1^{-1} \otimes S_1^{-1} \right) D_N \right)^{-1}$$

where $D_N$ is the duplication matrix (see Magnus and Neudecker, 1999) and $\otimes$ is the Kronecker product.
ROBUST BAYESIAN MV – optimal portfolios in practice

Under the above assumptions the robust Bayesian mean-variance problem:

\[ w_{p,q}^{(i)} \equiv \arg\max_{w \in C} \left\{ \min_{m \in \Theta_m} m' w \right\} \]

subject to \( \left\{ w' S w \right\} \leq v^{(i)} \)

…simplifies as follows:

\[ w_{p,q}^{(i)} \subset w(\lambda) \equiv \arg\max_{w \in C} \left\{ m_1' w - \lambda \sqrt{w' S_1 w} \right\} \]

• The **three-dimensional frontier** collapses to a **line**
• The efficient frontier is parametrized by the exposure to **overall risk**, which includes **market risk**, **estimation risk** for \( m \) and **estimation risk** for \( S \)
ROBUST BAYESIAN MV – Bayesian self-adjusting nature

• When the number of historical observations is large the uncertainty regions collapse to classical sample point estimates:

\[
\mathbf{w}(\lambda) \equiv \arg\max_{\mathbf{w} \in \mathcal{C}} \left\{ \mathbf{w}' \hat{\mathbf{m}}_1 - \lambda \sqrt{\mathbf{w}' \hat{\mathbf{S}}_1 \mathbf{w}} \right\}
\]

robust Bayesian frontier = classical sample-based frontier

• When the confidence in the prior is large the uncertainty regions collapse to the prior parameters:

\[
\mathbf{w}(\lambda) \equiv \arg\max_{\mathbf{w} \in \mathcal{C}} \left\{ \mathbf{w}' \mathbf{m}_0 - \lambda \sqrt{\mathbf{w}' \mathbf{S}_0 \mathbf{w}} \right\}
\]

robust Bayesian frontier = “a-priori” frontier (no information from the market)
ROBUST BAYESIAN MV – Bayesian self-adjusting nature

Robust Bayesian Frontier

$T \ll T_0, \nu_0$

Prior Frontier

$T \gg T_0, \nu_0$

Sample-Based Frontier

Portfolio weights

Market & Estimation Risk
A. Meucci

This encyclopedic, detailed exposition spans all the steps of one-period allocation from the foundations to the most advanced developments.

Multivariate estimation methods are analyzed in depth, including non-parametric, maximum-likelihood under non-normal hypotheses, shrinkage, robust, and very general Bayesian techniques. Evaluation methods such as stochastic dominance, expected utility, value at risk and coherent measures are thoroughly discussed in a unified setting and applied in a variety of contexts, including prospect theory, total return and benchmark allocation. Portfolio optimization is presented with emphasis on estimation risk, which is tackled by means of Bayesian, resampling and robust optimization techniques.

All the statistical and mathematical tools, such as copulas, location-dispersion ellipsoids, matrix-variate distributions, cone programming, are introduced from the basics.

At symmys.com the reader will find freely downloadable complementary materials: the Exercise Book; a set of thoroughly documented MATLAB® applications; and the Technical Appendices with all the proofs.

More materials and complete reviews can also be found at symmys.com.

This exciting new book takes a fresh look at asset allocation and offers up a masterly account of this important subject. The quantitative emphasis and included MATLAB software make it a must-read for the mathematically-oriented investment professional.

Peter Carr, Head of Quantitative Research, Bloomberg LP., Director of Masters in Mathematical Finance, NYU

Meucci’s Risk and Asset Allocation is one of those rare books that take a completely fresh look at a well-studied problem, optimal financial portfolio allocation based on statistically estimated models of risk and expected return. Designed for graduate students or quantitatively-oriented asset managers, Meucci provides a sophisticated and integrated treatment (...). This is rigorous and relevant!

Darrell Duffie, Professor of Finance, Stanford University

A wonderful book! Mathematically rigorous and yet practical, heavily illustrated with graphs and worked examples, Attilio Meucci has written a comprehensive treatment of asset allocation (...).

Bob Litterman, Head of Quantitative Resources, Goldman Sachs Asset Management

This book takes the reader on a journey through portfolio management starting with the basics and reaching some fascinating terrain. Attilio Meucci shows a real talent for explaining the most difficult of subjects in a very clear manner.

Paul Wilmott (wilmott.com)